

# ON THE STRUCTURE OF SUBSETS OF THE DISCRETE CUBE WITH SMALL EDGE BOUNDARY

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**ABSTRACT.** The edge isoperimetric inequality in the discrete cube specifies, for each pair of integers  $m$  and  $n$ , the minimum size  $g_n(m)$  of the edge boundary of an  $m$ -element subset of  $\{0, 1\}^n$ ; the extremal families (up to automorphisms of the discrete cube) are initial segments of the lexicographic ordering on  $\{0, 1\}^n$ . We show that for any  $m$ -element subset  $\mathcal{F} \subset \{0, 1\}^n$  and any integer  $l$ , if the edge boundary of  $\mathcal{F}$  has size at most  $g_n(m) + l$ , then there exists an extremal family  $\mathcal{G} \subset \{0, 1\}^n$  such that  $|\mathcal{F} \Delta \mathcal{G}| \leq Cl$ , where  $C$  is an absolute constant. This is best-possible, up to the value of  $C$ . Our result can be seen as a ‘stability’ version of the edge isoperimetric inequality in the discrete cube, and as a discrete analogue of the seminal stability result of Fusco, Maggi and Pratelli [14] for the isoperimetric inequality in Euclidean space.

## 1. INTRODUCTION

Isoperimetric inequalities are of ancient interest in mathematics. In general, an isoperimetric inequality gives a lower bound on the ‘boundary-size’ of a set of a given ‘size’, where the exact meaning of these words varies according to the problem. One of the best-known examples is the isoperimetric inequality for Euclidean space, which states (informally) that among all subsets of  $\mathbb{R}^n$  of given volume (whose surface area is defined), a Euclidean ball has the smallest surface area. An exact formulation (actually, one of the versions) of this inequality is as follows.

**Theorem 1.1.** *If  $A \subset \mathbb{R}^n$  is a Borel set with Lebesgue measure  $\mu(A) < \infty$ , then*

$$\text{Per}(A) \geq \text{Per}(B),$$

*where  $B$  is a Euclidean ball in  $\mathbb{R}^n$  with  $\mu(B) = \mu(A)$ .*

Here,  $\text{Per}(S)$  denotes the *distributional perimeter* of a set  $S \subset \mathbb{R}^n$ , which is equal to the  $(n - 1)$ -dimensional Lebesgue measure of the topological boundary of  $S$ , for sufficiently ‘nice’ sets  $S$ . (E.g., it suffices for  $S$  to be a Borel set with finite Lebesgue measure and piecewise smooth topological boundary.)

When an isoperimetric inequality is sharp, and the extremal sets are known, it is natural to ask whether the inequality is also ‘stable’ — i.e., if a set has boundary of size ‘close’ to the minimum, must that set be ‘close in structure’ to an extremal set?

In their seminal work [14], Fusco, Maggi and Pratelli obtained a stability result for Theorem 1.1, confirming a conjecture of Hall [15].

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**Theorem 1.2** (Fusco, Maggi, Pratelli, 2008). *Let  $\epsilon > 0$ . Suppose  $A \subset \mathbb{R}^n$  is a Borel set with Lebesgue measure  $\mu(A) < \infty$ , and with*

$$\text{Per}(A) \leq (1 + \epsilon)\text{Per}(B),$$

*where  $B$  is a Euclidean ball with  $\mu(B) = \mu(A)$ . Then there exists  $x \in \mathbb{R}^n$  such that*

$$\mu(A \Delta (B + x)) \leq C_n \sqrt{\epsilon} \mu(A),$$

*where  $C_n > 0$  is a constant depending upon  $n$  alone.*

As observed in [14], Theorem 1.2 is sharp up to the value of the constant  $C_n$ , as can be seen by taking  $S$  to be an ellipsoid with  $n - 1$  semi-axes of length 1 and one semi-axis of length slightly larger than 1.

### Discrete isoperimetric inequalities

In the last fifty years, there has been a great deal of interest in *discrete* isoperimetric inequalities. These deal with the boundaries of sets of vertices in graphs. There are two natural measures of the boundary of a set of vertices in a graph: the *edge boundary* and the *vertex boundary*. If  $G = (V, E)$  is a graph, and  $A \subset V$  is a set of vertices of  $G$ , the *edge boundary* of  $A$  consists of the set of edges of  $G$  which join a vertex in  $A$  to a vertex in  $V \setminus A$ ; it is denoted by  $\partial_G(A)$ , or by  $\partial A$  when the graph  $G$  is understood. The *vertex boundary* of  $A$  consists of the set of vertices of  $V \setminus A$  which are adjacent to a vertex in  $A$ ; it is sometimes denoted by  $b_G(A)$ , or by  $b(A)$  when the graph  $G$  is understood. If  $G = (V, E)$  is a graph, the *edge isoperimetric problem for  $G$*  asks for a determination of  $\Phi_G(m) := \min\{|\partial A| : A \subset V, |A| = m\}$ , for each integer  $m$ ; similarly, the *vertex isoperimetric problem for  $G$*  asks for a determination of  $\Psi_G(m) := \min\{|b(A)| : A \subset V, |A| = m\}$ , for each integer  $m$ .

An important example of a discrete isoperimetric problem, and the focus of this paper, is the edge-isoperimetric problem for the  $n$ -dimensional discrete cube,  $Q_n$ . (We define  $Q_n$  to be the graph with vertex-set  $\{0, 1\}^n$ , where two 0-1 vectors are adjacent if they differ in exactly one coordinate.) This isoperimetric problem has numerous applications, both to other problems in mathematics, and in other areas such as communication complexity (see e.g. [16]), network science (see [3]) and game theory (see [17]). Hereafter, if  $A \subset \{0, 1\}^n$ , we write  $\partial A$  for the edge boundary of  $A$  with respect to  $Q_n$ .

The edge isoperimetric problem for  $Q_n$  was solved by Harper [16], Lindsay [26], Bernstein [2], and Hart [17]. Let us describe the solution. We may identify  $\{0, 1\}^n$  with the power-set  $\mathcal{P}([n])$  of  $[n] := \{1, 2, \dots, n\}$ , by identifying a 0-1 vector  $(x_1, \dots, x_n)$  with the set  $\{i \in [n] : x_i = 1\}$ . We can then view  $Q_n$  as the graph with vertex set  $\mathcal{P}([n])$ , where two sets  $S, T \subset [n]$  are adjacent if  $|S \Delta T| = 1$ . The *lexicographic ordering* on  $\mathcal{P}([n])$  is defined by  $S > T$  iff  $\min(S \Delta T) \in S$ . If  $m \in [2^n]$ , the *initial segment of the lexicographic ordering on  $\mathcal{P}([n])$  of size  $m$*  is simply the subset of  $\mathcal{P}([n])$  consisting of the  $m$  largest elements of  $\mathcal{P}([n])$  with respect to the lexicographic ordering. If  $\mathcal{L} \subset \mathcal{P}([n])$  is an initial segment of the lexicographic ordering, we say  $\mathcal{L}$  is *lexicographically ordered*. Note that if  $m = 2^d$  for some  $d \in \mathbb{N}$ , then the initial segment of the lexicographic ordering on  $\mathcal{P}([n])$  of size  $m$  is the  $d$ -dimensional subcube  $\{S \subset [n] : [n - d] \subset S\}$ .

Harper, Bernstein, Lindsay and Hart proved the following.

**Theorem 1.3** (The edge isoperimetric inequality for  $Q_n$ ). *If  $\mathcal{F} \subset \mathcal{P}([n])$ , then  $|\partial \mathcal{F}| \geq |\partial \mathcal{L}|$ , where  $\mathcal{L} \subset \mathcal{P}([n])$  is the initial segment of the lexicographic ordering of size  $|\mathcal{F}|$ .*

Let us describe the extremal families for Theorem 1.3. If  $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$ , we say that  $\mathcal{F}$  and  $\mathcal{G}$  are *weakly isomorphic* if there exists an automorphism  $\phi$  of  $Q_n$  such that  $\mathcal{G} = \phi(\mathcal{F})$ ; in this case, we write  $\mathcal{F} \cong \mathcal{G}$ . Equivalently,  $\mathcal{F}, \mathcal{G} \subset \{0, 1\}^n$  are weakly isomorphic iff  $\mathcal{G}$  can be obtained from  $\mathcal{F}$  by permuting the coordinates  $1, 2, \dots, n$  and interchanging 0's with 1's on some subset of the coordinates. Clearly, weak isomorphism preserves the size of the edge boundary. It is well-known (and easy to check by analyzing known proofs of Theorem 1.3) that equality holds in Theorem 1.3 if and only if  $\mathcal{F}$  is weakly isomorphic to  $\mathcal{L}$ . In particular, if  $|\mathcal{F}|$  is a power of 2, then equality holds in Theorem 1.3 if and only if  $\mathcal{F}$  is a subcube.

To date, several stability versions of Theorem 1.3 have been obtained. Using a Fourier-analytic argument, Friedgut, Kalai and Naor [13] obtained a stability result for sets of size  $2^{n-1}$ , showing that if  $\mathcal{F} \subset \mathcal{P}([n])$  with  $|\mathcal{F}| = 2^{n-1}$  and  $|\partial\mathcal{F}| \leq (1 + \epsilon)2^{n-1}$ , then  $|\mathcal{F}\Delta\mathcal{C}|/2^n = O(\epsilon)$  for some codimension-1 subcube  $\mathcal{C}$ . (The dependence upon  $\epsilon$  here is almost sharp, viz., sharp up to a factor of  $\Theta(\log(1/\epsilon))$ ). Bollobás, Leader and Riordan (unpublished) proved an analogous result for  $|\mathcal{F}| \in \{2^{n-2}, 2^{n-3}\}$ , also using a Fourier-analytic argument. Samorodnitsky [27] used a result of Keevash [22] on the structure of  $r$ -uniform hypergraphs with small shadows, to prove a stability result for all  $\mathcal{F} \subset \mathcal{P}([n])$  with  $\log_2 |\mathcal{F}| \in \mathbb{N}$ , under the rather strong condition  $|\partial\mathcal{F}| \leq (1 + O(1/n^4))|\partial\mathcal{L}|$ . In [6], the first author proved the following stability result (which implies the above results), using a recursive approach and an inequality of Talagrand [28] (which is proved via Fourier analysis).

**Theorem 1.4** ([6]). *There exists an absolute constant  $c > 0$  such that the following holds. Let  $0 \leq \delta < c$ . If  $\mathcal{F} \subset \mathcal{P}([n])$  with  $|\mathcal{F}| = 2^d$  for some  $d \in \mathbb{N}$ , and  $|\mathcal{F}\Delta\mathcal{C}| \geq \delta 2^d$  for all  $d$ -dimensional subcubes  $\mathcal{C} \subset \mathcal{P}([n])$ , then*

$$|\partial\mathcal{F}| \geq |\partial\mathcal{C}| + 2^d \delta \log_2(1/\delta).$$

As observed in [6], this result is best-possible (except for the condition  $0 \leq \delta < c$ , which was conjectured to be unnecessary in [6]). However, the problem of obtaining a sharp stability result for sets not of size a power of 2, remained open.

### Our result

In this paper, we obtain the following stability result for the edge isoperimetric inequality in the discrete cube, which applies to families of arbitrary size and which is sharp up to an absolute constant factor.

**Theorem 1.5.** *There exists an absolute constant  $C > 0$  such that the following holds. If  $\mathcal{F} \subset \mathcal{P}([n])$  and  $\mathcal{L} \subset \mathcal{P}([n])$  is the initial segment of the lexicographic ordering with  $|\mathcal{L}| = |\mathcal{F}|$ , then there exists a family  $\mathcal{G} \subset \mathcal{P}([n])$  weakly isomorphic to  $\mathcal{L}$ , such that*

$$|\mathcal{F}\Delta\mathcal{G}| \leq C(|\partial\mathcal{F}| - |\partial\mathcal{L}|).$$

This is sharp up to the value of the absolute constant  $C$ . In fact, we conjecture that Theorem 1.5 holds with  $C = 2$ , due to the following example.

**Example 1.6.** Let  $s, t, n$  be integers with  $t \geq 2$  and  $t + 2 \leq s \leq n$ , and let

$$\mathcal{F} = \mathcal{F}_{n,s,t} = \{S \subset [n] : [t] \subset S\} \cup \{S \subset [n] : \{1, 2\} \not\subset S, \{3, \dots, s\} \subset S\}.$$

It is easy to see that

$$\min\{|\mathcal{F}\Delta\mathcal{G}| : \mathcal{G} \cong \mathcal{L}, \mathcal{L} \text{ is an initial segment of lex}, |\mathcal{L}| = |\mathcal{F}|\} = 2(|\partial\mathcal{F}| - |\partial\mathcal{L}|),$$

for each of the above families  $\mathcal{F}$ .

We note that the relation between  $|\partial\mathcal{F}| - |\partial\mathcal{L}|$  and  $|\mathcal{F}\Delta\mathcal{G}|$  in Theorem 1.4 (which applies in the special case where  $|\mathcal{F}|$  is a power of 2) is sharper than in Theorem 1.5, but the above example demonstrates that Theorem 1.5 is sharp (up to an absolute constant factor) in its more general setting.

Instead of the Fourier-analytic techniques used in most previous works on isoperimetric stability, our techniques are purely combinatorial. As is often the case with theorems concerning  $Q_n$ , we prove Theorem 1.5 by induction on  $n$ , but the techniques we use in the inductive step are somewhat novel. The inductive step relies on an ‘intermediate’ structure theorem (Proposition 4.1) concerning the intersections of  $\mathcal{F}$  with codimension-1 and codimension-2 subcubes, where  $\mathcal{F} \subset \mathcal{P}([n])$  is a family with small edge boundary. This proposition is proved using some intricate combinatorial arguments, including shifting operators (a.k.a. ‘compressions’), and a detailed analysis of the *influences* of the family (see below).

### Related work

The edge boundary  $\partial\mathcal{F}$  of a subset  $\mathcal{F} \subset \mathcal{P}([n])$  is closely connected with the *influences* of  $\mathcal{F}$ . For  $\mathcal{F} \subset \mathcal{P}([n])$ , the  $i$ th influence of  $\mathcal{F}$  is defined by

$$\text{Inf}_i[\mathcal{F}] := |\{A \subset [n] : |\mathcal{F} \cap \{A, A\Delta\{i\}\}| = 1\}|/2^n,$$

and the total influence of  $\mathcal{F}$  is  $I[\mathcal{F}] := \sum_{i=1}^n \text{Inf}_i[\mathcal{F}]$ . Note that  $I[\mathcal{F}] = |\partial\mathcal{F}|/2^{n-1}$  — the total influence of a set is none other than the size of its edge boundary, appropriately normalised.

It is natural to rephrase this definition in the language of Boolean functions. If  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ , we define the  $i$ th influence of  $f$  to be the probability that, if  $x \in \{0, 1\}^n$  is chosen uniformly at random and the  $i$ th entry of  $x$  is flipped, the value of the function  $f$  changes. There is of course a natural one-to-one correspondence between subsets of  $\mathcal{P}([n])$  and Boolean functions on  $\{0, 1\}^n$ : for each  $\mathcal{F} \subset \mathcal{P}([n])$ , we associate to  $\mathcal{F}$  the Boolean function  $\chi_{\mathcal{F}} : \{0, 1\}^n \rightarrow \{0, 1\}$  defined by  $\chi_{\mathcal{F}}(x) = 1$  iff  $\{i \in [n] : x_i = 1\} \in \mathcal{F}$ ; the  $i$ th influence of  $\mathcal{F}$  is then precisely the  $i$ th influence of  $\chi_{\mathcal{F}}$ .

Over the last thirty years, many results have been obtained on the influences of Boolean functions (and functions on more general product spaces), and have proved extremely useful in such diverse fields as theoretical computer science, social choice theory and statistical physics, as well as in combinatorics (see, e.g., the survey [21]). One of the most useful such results (and one of the first major results on influences) is the seminal ‘KKL theorem’ (Kahn, Kalai and Linial [20]), which states that for any Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  with  $\mathbb{E}[f] = \mu$ , there exists  $i \in [n]$  such that

$$\text{Inf}_i[f] \geq c_0 \mu(1 - \mu) \frac{\log n}{n},$$

where  $c_0$  is an absolute constant — so a Boolean function of expectation  $1/2$  has some coordinate of ‘fairly large’ influence, viz.,  $\Omega((\log n)/n)$ . (Note that if  $\mathbb{E}[f] = 1/2$ , then Theorem 1.3 implies that  $I[f] \geq 1$ , so a naive averaging argument only supplies a coordinate of influence at least  $1/n$ .) The proof of Kahn, Kalai and Linial made crucial use of Fourier analysis on the discrete cube, together with the hypercontractive inequality due (independently) to Gross, Bonami and Beckner. Another very useful example is Friedgut’s ‘Junta’ theorem [10]:

**Theorem 1.7** (Friedgut’s Junta theorem). *There exists an absolute constant  $C$  such that the following holds. Let  $\epsilon > 0$ , and let  $\mathcal{F} \subset \mathcal{P}([n])$ . Then there exists  $\mathcal{G} \subset \mathcal{P}([n])$  depending upon at most  $2^{CI[\mathcal{F}]/\epsilon}$  coordinates, such that  $|\mathcal{F} \Delta \mathcal{G}| \leq \epsilon 2^n$ .*

Here, to be completely formal, if  $\mathcal{G} \subset \mathcal{P}([n])$  we say that  $\mathcal{G}$  depends upon  $k$  coordinates if there exists  $S \subset [n]$  with  $|S| = k$ , such that  $(T \in \mathcal{G}) \Leftrightarrow (T \cap S \in \mathcal{G})$  holds for all  $T \subset [n]$ . Friedgut’s theorem implies that any  $\mathcal{F} \subset \mathcal{P}([n])$  with bounded total influence (at most  $K$ , say), and with measure  $|\mathcal{F}|/2^n$  bounded away from 0 and 1, can be closely approximated by a ‘junta’ — that is, by a family which depends upon a bounded number of coordinates (depending on  $K$ ). Friedgut’s proof in [10] uses Fourier analysis and hypercontractivity, in a similar way to the proof of the KKL theorem.

For families  $\mathcal{F} \subset \mathcal{P}([n])$  with measure  $|\mathcal{F}|/2^n \in [\alpha, 1 - \alpha]$ , Theorem 1.3 implies that  $I[\mathcal{F}] \geq 2\alpha \log_2(1/\alpha)$ . Hence, Friedgut’s theorem can be viewed as a structure theorem for families of measure bounded away from 0 and 1, whose total influence lies within a constant multiplicative factor of the minimum possible total influence. Similarly, Friedgut [11], Bourgain [4] and Hatami [18] obtained structure theorems for ‘large’ subsets of  $\mathcal{P}([n])$  whose ‘biased’ measure lies within a constant multiplicative factor of the minimum possible, and Kahn and Kalai [19] stated several conjectures on ‘small’ subsets of  $\mathcal{P}([n])$  satisfying the same condition. The results of [4, 11, 18] are deep, with many important applications.

In contrast to the results of [4, 11, 18], which describe the structure of families with total influence within a constant factor of the minimum, our Theorem 1.5 describes the structure of Boolean functions with total influence ‘very’ close to the minimum. On the other hand, the structure we obtain is very strong — namely, closeness to a genuinely extremal family.

## 2. OUTLINE OF THE PROOF AND ORGANIZATION OF THE PAPER

The main step of our proof is showing an ‘intermediate’ structural result (Proposition 4.1 below) for families of small edge boundary (i.e., small total influence). Informally, this says that if  $\mathcal{F} \subset \mathcal{P}([n])$  such that  $|\mathcal{F}| \leq 2^{n-1}$  and  $I[\mathcal{F}] \leq I[\mathcal{L}] + \epsilon$ , where  $\mathcal{L}$  the initial segment of the lexicographic ordering on  $\mathcal{P}([n])$  of size  $|\mathcal{F}|$  and  $\epsilon$  is sufficiently small, then one of the following must hold.

- Case (1):  $\mathcal{F}$  is essentially contained in a subcube of codimension 1 (i.e., in a family depending upon just one coordinate), and the total influence of the part of  $\mathcal{F}$  inside the subcube is ‘small’.
- Case (2):  $\mathcal{F}$  is essentially contained in a subcube of codimension 2 (i.e., in a family depending upon just two coordinates), and the total influence of the part of  $\mathcal{F}$  inside that subcube is ‘small’.

Once Proposition 4.1 is established, the main theorem follows by a short induction on  $n$  (Proposition 4.1 is needed for the inductive step). It is perhaps fortunate that, for the inductive step, it suffices to pass to subcubes of codimension at most 2. Interestingly, it does not suffice to pass to subcubes of codimension 1, as we explain in Section 4; this might, at first glance, deceive one into abandoning an inductive approach.

The proof of Proposition 4.1 is divided into two parts. In the first part, we prove that if  $|\mathcal{F}|$  is ‘sufficiently large’ (specifically, if  $|\mathcal{F}| \geq 2^{n-2}(1+c)$  for an absolute constant  $c > 0$ ), then  $\mathcal{F}$  must satisfy (1); this is the content of Proposition

7.1. In the second part, we prove that if  $|\mathcal{F}| < 2^{n-2}(1+c)$ , then  $\mathcal{F}$  must satisfy (2); this is the content of Proposition 8.1. The harder part is the first one; the proof is (again) by induction on  $n$ , but with six ingredients that are outlined at the beginning of Section 7. Roughly speaking, we define a collection of ‘small alterations’ which preserve the property of being a counterexample to Proposition 7.1; applying a sequence of these small alterations, we reduce to the case where the family is sufficiently ‘well-behaved’ for us to successfully apply the inductive hypothesis. (Note that a very similar technique was used in [23].) The second part uses the classical *shifting* technique [5, 7]: we first reduce to the case where  $\mathcal{F}$  is monotone increasing; we then choose the coordinate of largest influence ( $i$  say), and apply appropriate shifting operators to  $\mathcal{F}$  to produce a family contained entirely within the codimension-1 subcube  $\{S \subset [n] : i \in S\}$ ; passing to this subcube, we obtain a family of twice the measure of the original family; we then repeat this process until the family is large enough that we can apply Proposition 7.1 (from the first part).

An important component of the proof of Proposition 4.1 is a pair of ‘bootstrapping’ lemmas showing that if  $\mathcal{F}$  is ‘somewhat’ close to being contained in a subcube of codimension 1 or 2, then it must be ‘very’ close to that subcube. In order to prove these bootstrapping lemmas, we introduce the notion of *fractional lexicographic families*, as a convenient technical tool. These allow us to analyse how the measure (or ‘mass’) of a family of small total influence can be distributed between two disjoint codimension-1 subcubes (or between four disjoint codimension-2 subcubes); informally, this distribution cannot differ too much from in the extremal, lexicographically ordered family  $\mathcal{L}$ .

### Organization of the paper

In Section 3, we introduce some definitions and notation, and present some basic facts on influences and shifting. In Section 4, we reduce the main theorem to the intermediate structural result, Proposition 4.1, discussed above. Fractional lexicographic families and their properties are studied in Section 5, and the bootstrapping lemmas are presented in Section 6.

The proof of Proposition 4.1 spans Sections 7-9. The case of ‘large’ families is covered in Section 7, ‘small’ families are dealt with in Section 8, and finally we combine these two cases to prove Proposition 4.1 in Section 9. We conclude with some open problems in Section 10.

## 3. PRELIMINARIES

**3.1. Notation.** We equip  $\mathcal{P}([n])$  with the uniform measure, denoted by  $\mu$ :

$$\mu(\mathcal{F}) = \frac{|\mathcal{F}|}{2^n} \quad \forall \mathcal{F} \subset \mathcal{P}([n]).$$

We write  $S \sim \mathcal{P}([n])$  to mean that  $S$  is chosen uniformly at randomly from  $\mathcal{P}([n])$ .

If  $C \subset B \subset [n]$ , and  $\mathcal{F} \subset \mathcal{P}([n])$ , we define the ‘sliced’ family

$$\mathcal{F}_B^C := \{S \setminus C : S \in \mathcal{F}, S \cap B = C\} \subset \mathcal{P}([n] \setminus B).$$

Note that we view  $\mathcal{F}_B^C$  as a subfamily of  $\mathcal{P}([n] \setminus B)$ , and so  $\mu(\mathcal{F}_B^C) = \frac{|\mathcal{F}_B^C|}{2^{n-|B|}}$ .

If  $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$ , we say that  $\mathcal{F}$  and  $\mathcal{G}$  are *weakly isomorphic* if there exists an automorphism  $\phi$  of  $Q_n$  such that  $\mathcal{G} = \phi(\mathcal{F})$ ; in this case, we write  $\mathcal{F} \cong \mathcal{G}$ .

To be completely formal and explicit, if  $\pi \in \text{Sym}([n])$  and  $S \subset [n]$ , we write  $\pi(S) := \{\pi(i) : i \in S\}$ , and if  $\mathcal{F} \subset \mathcal{P}([n])$ , we write  $\pi(\mathcal{F}) := \{\pi(S) : S \in \mathcal{F}\}$ . Families  $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$  are said to be *isomorphic* if there exists  $\pi \in \text{Sym}([n])$  such that  $\mathcal{G} = \pi(\mathcal{F})$ . If  $D \subset [n]$  and  $\mathcal{F} \subset \mathcal{P}([n])$ , we define  $X_D(\mathcal{F}) = \{S \Delta D : S \in \mathcal{F}\}$ . Families  $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$  are weakly isomorphic iff there exist  $\pi \in \text{Sym}([n])$  and  $D \subset [n]$  such that  $\mathcal{G} = X_D(\pi(\mathcal{F}))$ .

For  $n \in \mathbb{N}$  and  $2^n \mu \in \{0, 1, \dots, 2^n\}$ , we let  $\mathcal{L}_{\mu, n}$  denote the initial segment of the lexicographic ordering on  $\mathcal{P}([n])$  with measure  $\mu$ . We write  $\mathbb{L}_{\mu, n}$  for the class of all families weakly isomorphic to  $\mathcal{L}_{\mu, n}$ . When  $n$  is understood, we will write these as  $\mathcal{L}_\mu$  and  $\mathbb{L}_\mu$ , suppressing the subscript  $n$ . If  $\mathcal{F} \subset \mathcal{P}([n])$ , we write

$$\mu(\mathcal{F} \Delta \mathbb{L}_\mu) := \min\{\mu(\mathcal{F} \Delta \mathcal{G}) : \mathcal{G} \cong \mathcal{L}_\mu\}.$$

We write  $\mu_i^- = \mu_i^-(\mathcal{F})$  for the measure  $\mu(\mathcal{F}_{\{i\}}^\emptyset)$ , and we write  $\mu_i^+ = \mu_i^+(\mathcal{F})$  for the measure  $\mu(\mathcal{F}_{\{i\}}^{\{i\}})$ . By the isoperimetric inequality (Theorem 1.3), we may write  $I[\mathcal{F}_{\{i\}}^{\{i\}}] = I[\mathcal{L}_{\mu_i^+}^{\{i\}}] + \epsilon_i^+$ , where  $\epsilon_i^+ = \epsilon_i^+(\mathcal{F}) \geq 0$ , and we use the notations

$$\mu_{i,j}^{++}, \mu_{i,j}^{+-}, \mu_{i,j}^{-+}, \mu_{i,j}^{--}, \epsilon_i^+, \epsilon_{i,j}^{++}, \epsilon_{i,j}^{+-}, \epsilon_{i,j}^{-+}, \epsilon_{i,j}^{--},$$

defined similarly. For  $B \subset [n]$ , we write  $\mathcal{S}_B := \{S \subset [n] : B \subset S\}$  for the subcube of all subsets of  $[n]$  that contain  $B$ , and if  $C \subset B$ , we write  $\mathcal{S}_B^C := \{S \subset [n] : S \cap B = C\}$  for the subcube of all subsets of  $[n]$  that intersect  $B$  on the set  $C$ . If  $j \in [n]$ , we write  $\mathcal{D}_j := \{S \subset [n] : j \in S\}$  for the ‘dictatorship’ consisting of all sets containing  $j$ .

We say that a family  $\mathcal{L} \subset \mathcal{P}([n])$  is *lexicographically ordered* if it is an initial segment of the lexicographic ordering on  $\mathcal{P}([n])$ . We say that a family  $\mathcal{F} \subset \mathcal{P}([n])$  is *monotone increasing* (or just *increasing*) if it is closed under taking supersets, i.e. whenever  $A \subset B \subset [n]$  and  $A \in \mathcal{F}$ , we have  $B \in \mathcal{F}$ .

**3.2. Influences.** Using the notation above, we may define the  $i$ th influence of a family  $\mathcal{F} \subset \mathcal{P}([n])$  by

$$\text{Inf}_i[\mathcal{F}] = \Pr_{A \sim \mathcal{P}([n])} [|\mathcal{F} \cap \{A, A \Delta \{i\}\}| = 1].$$

As mentioned in the introduction, we have

$$I[\mathcal{F}] = \sum_{i=1}^n \text{Inf}_i[\mathcal{F}] = \frac{|\partial \mathcal{F}|}{2^{n-1}},$$

i.e. the total influence of  $\mathcal{F}$  is the normalized edge boundary of  $\mathcal{F}$ . We may therefore restate our main theorem (Theorem 1.5) as follows.

**Theorem.** *There exists an absolute constant  $C > 0$  such that the following holds. Let  $\epsilon > 0$ , let  $\mathcal{F} \subset \mathcal{P}([n])$  be a family of measure  $\mu$ , and suppose that  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + \epsilon$ . Then there exists a family  $\mathcal{G} \subset \mathcal{P}([n])$  weakly isomorphic to  $\mathcal{L}_\mu$ , such that  $\mu(\mathcal{F} \Delta \mathcal{G}) \leq C\epsilon$ .*

(Note that the constant  $C$  above is half the constant in the original statement.) It will be more convenient for us to work with the above reformulation.

If  $\mathcal{F} \subset \mathcal{P}([n])$  and  $i \in [n]$ , we define the family of  *$i$ -pivotal sets in  $\mathcal{F}$*  by

$$\mathcal{I}_i(\mathcal{F}) := \{A \in \mathcal{F} : A \Delta \{i\} \notin \mathcal{F}\}.$$

Note that we have

$$\text{Inf}_i [\mathcal{F}] = \frac{|\mathcal{I}_i(\mathcal{F})|}{2^{n-1}}$$

for all  $i \in [n]$ .

The following lemma will be useful for relating the influence of a family  $\mathcal{F}$  to the influences of its slices.

**Lemma 3.1.** *If  $S \subset [n]$  and  $\mathcal{F} \subset \mathcal{P}([n])$ , then*

$$\begin{aligned} I[\mathcal{F}] &= \frac{1}{2^{|S|}} \sum_{B \subset S} I[\mathcal{F}_S^B] + \frac{1}{2^{|S|-1}} \sum_{B \subset S} \sum_{i \in B} \mu(\mathcal{F}_S^B \Delta \mathcal{F}_S^{B \setminus \{i\}}) \\ &= \mathbb{E}_{B \sim \mathcal{P}(S)} I[\mathcal{F}_S^B] + \sum_{i \in S} \text{Inf}_i [\mathcal{F}]. \end{aligned}$$

The proof is straightforward, and we omit it.

**3.3. Shifting.** The following shifting operator  $\mathcal{S}_{S,T}$  was introduced by Erdős, Ko and Rado [7] in the case  $|S| = |T| = 1$ ; for larger values of  $|S|$  or  $|T|$ , it was introduced by Daykin [5].

**Definition 3.2.** Let  $n \in \mathbb{N}$ , let  $\mathcal{F} \subset \mathcal{P}([n])$ , and let  $S, T \subset [n]$  with  $S \cap T = \emptyset$ . For a set  $A \in \mathcal{F}$ , we define

$$\mathcal{S}_{ST}(A) := \begin{cases} (A \setminus S) \cup T & \text{if } S \subset A, A \cap T = \emptyset \text{ and } (A \setminus S) \cup T \notin \mathcal{F}, \\ A & \text{otherwise.} \end{cases}$$

We define  $\mathcal{S}_{ST}(\mathcal{F}) := \{\mathcal{S}_{ST}(A) : A \in \mathcal{F}\}$ .

Observe that  $\mathcal{S}_{ST}(\mathcal{F})$  is the family  $\mathcal{G} \subset \mathcal{P}([n])$  such that  $\mathcal{G}_{S \cup T}^B = \mathcal{F}_{S \cup T}^B$  for any  $B \neq S, T$ , such that  $\mathcal{G}_{S \cup T}^S = \mathcal{F}_{S \cup T}^S \cap \mathcal{F}_{S \cup T}^T$ , and such that  $\mathcal{G}_{S \cup T}^T = \mathcal{F}_{S \cup T}^S \cup \mathcal{F}_{S \cup T}^T$ .

These shifting operators are known to be a very useful tool in extremal combinatorics. They were used by Frankl [9] to obtain stability results for the Erdős-Ko-Rado theorem [7], and were recently applied by the second and third authors in [23] to obtain a stability result for the Ahlswede-Khachatrian theorem [1], thus proving a conjecture of Friedgut [12]. A major part of our argument is based on the method of [23].

The following lemma says that if a family  $\mathcal{F}$  is stable under ‘lower-order’ shifts, then a shifting operator cannot increase the total influence of  $\mathcal{F}$ .

**Lemma 3.3.** *Let  $\mathcal{F} \subset \mathcal{P}([n])$ , and let  $S, T \subset [n]$  with  $S \cap T = \emptyset$  and  $|S| \geq |T|$ . Suppose that  $\mathcal{S}_{S'T}(\mathcal{F}) = \mathcal{F}$  for each  $S' \subset S$  with  $|S'| = |S| - 1$ . Then  $I[\mathcal{S}_{ST}(\mathcal{F})] \leq I[\mathcal{F}]$ .*

*Proof.* Write  $\mathcal{G} = \mathcal{S}_{ST}(\mathcal{F})$ . By Lemma 3.1, we have

$$I[\mathcal{F}] = \sum_{i \in [n] \setminus (S \cup T)} \text{Inf}_i [\mathcal{F}] + \mathbb{E}_{B \sim \mathcal{P}([n] \setminus (S \cup T))} I[\mathcal{F}_{[n] \setminus (S \cup T)}^B]$$

and

$$I[\mathcal{G}] = \sum_{i \in [n] \setminus (S \cup T)} \text{Inf}_i [\mathcal{G}] + \mathbb{E}_{B \sim \mathcal{P}([n] \setminus (S \cup T))} I[\mathcal{G}_{[n] \setminus (S \cup T)}^B].$$

To prove the claim, it suffices to show that for any family  $\mathcal{F} \subset \mathcal{P}([n])$  and any  $i \notin S \cup T$ , we have  $\text{Inf}_i [\mathcal{F}] \geq \text{Inf}_i [\mathcal{S}_{ST}(\mathcal{F})]$ , and that for any family  $\mathcal{F} \subset \mathcal{P}(S \cup T)$  such that  $\mathcal{S}_{S'T}(\mathcal{F}) = \mathcal{F}$  for each  $S' \subset S$  with  $|S'| = |S| - 1$ , we have  $I[\mathcal{S}_{ST}(\mathcal{F})] \leq I[\mathcal{F}]$ .



$I[\mathcal{F}]$ . The verification of the former assertion is straightforward, and we leave it to the reader.

To prove the latter assertion, we may assume that  $\mathcal{S}_{ST}(\mathcal{F}) \neq \mathcal{F}$ ; then  $S \in \mathcal{F}$ ,  $T \notin \mathcal{F}$  and  $\mathcal{S}_{ST}(\mathcal{F}) = (\mathcal{F} \setminus \{S\}) \cup \{T\}$ . Note that  $S' \notin \mathcal{F}$  for all  $S' \subset S$  with  $|S'| = |S| - 1$ , and that  $T' \in \mathcal{F}$  for any  $T' \supset T$  with  $|T'| = |T| + 1$ . (Indeed, if  $S' \subset S$  with  $|S'| = |S| - 1$  and  $S' \in \mathcal{F}$ , then we have  $T = \mathcal{S}_{S'T}(S') \in \mathcal{F}$ , contradicting our assumption. Similarly, for each  $T' \supset T$  with  $|T'| = |T| + 1$ , we have  $T' = \mathcal{S}_{([S \cup T] \setminus T')T}(S) \in \mathcal{F}$ .) Therefore,  $|\partial(\mathcal{S}_{ST}(\mathcal{F}))| \leq |\partial\mathcal{F}| - 2(|S| - |T|) \leq |\partial\mathcal{F}|$ , as required.  $\square$

We also need the following well-known lemma on the so-called ‘monotonization operators’  $\mathcal{S}_{\emptyset\{i\}}$  (see, e.g., [20]).

**Lemma 3.4.** *Let  $i \in [n]$  and let  $\mathcal{F} \subset \mathcal{P}([n])$ . Then*

$$\inf_j [\mathcal{S}_{\emptyset\{i\}}(\mathcal{F})] \leq \inf_j [\mathcal{F}]$$

*for each  $j \in [n]$ , and  $I[\mathcal{S}_{\emptyset\{i\}}(\mathcal{F})] \leq I[\mathcal{F}]$ .*

#### 4. REDUCTION OF THEOREM 1.5

In this section we reduce Theorem 1.5 to the following proposition.

**Proposition 4.1.** *There exist absolute constants  $c_1, c_2 > 0$  such that the following holds. Let  $0 < \mu \leq \frac{1}{2}$ , let  $0 \leq \epsilon \leq c_1\mu$ , and let  $\mathcal{F} \subset \mathcal{P}([n])$  be a family with  $\mu(\mathcal{F}) = \mu$  and  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + \epsilon$ . Then there exists a family  $\mathcal{G}$  weakly isomorphic to  $\mathcal{F}$  such that one of the following holds.*

- *Case (1):  $c_2\mu_1^-(\mathcal{G}) + \frac{1}{2}\epsilon_1^+(\mathcal{G}) \leq \epsilon$ , or else*
- *Case (2):  $c_2\mu(\mathcal{G} \setminus \mathcal{S}_{\{1,2\}}) + \frac{1}{4}\epsilon_{1,2}^{++}(\mathcal{G}) \leq \epsilon$ .*

Intuitively, Proposition 4.1 says that there is a family  $\mathcal{G}$  weakly isomorphic to  $\mathcal{F}$ , such that one of the following holds: either (1)  $\mathcal{G}$  is essentially contained in the dictatorship  $\mathcal{D}_1$ , and the ‘essential part’  $\mathcal{G}_{\{1\}}^{\{1\}}$  has small total influence, or (2)  $\mathcal{G}$  is essentially contained in the subcube  $\mathcal{S}_{\{1,2\}}$ , and the ‘essential part’  $\mathcal{G}_{\{1,2\}}^{\{1,2\}}$  has small total influence.

Case (1) is in a sense the ‘simpler’ case, and it is natural to ask whether Case (2) can be removed; but it cannot. Indeed, for any  $c_1, c_2 > 0$ , if  $t$  is sufficiently large depending on  $c_1, c_2$  then the family

$$\mathcal{F} = \{S \subset [n] : \{1, 2\} \subset S, S \cap \{3, 4, \dots, t\} \neq \emptyset\} \cup \{S \subset [n] : \{3, 4, \dots, t\} \subset S\}$$

satisfies the hypotheses of Proposition 4.1, and Case (1) does not occur for  $\mathcal{F}$ .

We now show how to deduce Theorem 1.5 from Proposition 4.1. We state Theorem 1.5 again below (in the influence form), for the convenience of the reader.

**Theorem.** *There exists an absolute constant  $C > 0$  such that the following holds. Let  $\mathcal{F} \subset \mathcal{P}([n])$  be a family of measure  $\mu(\mathcal{F}) = \mu$ , and suppose that  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + \epsilon$ . Then there exists a family  $\mathcal{G} \subset \mathcal{P}([n])$  weakly isomorphic to  $\mathcal{L}_\mu$ , such that  $\mu(\mathcal{F} \Delta \mathcal{G}) \leq C\epsilon$ .*

*Proof.* We prove the theorem by induction on  $n$ . If  $n = 1$ , then  $\mathcal{F}$  itself is weakly isomorphic to  $\mathcal{L}_\mu$ . Let  $n \geq 2$ , and assume the statement of the theorem holds for smaller values of  $n$ .

We now make several reductions. Firstly, we note that the theorem holds for  $\mathcal{F}$  if and only if it holds for its complement  $\mathcal{F}^c$ , since the complement of a lexicographically ordered family is weakly isomorphic to a lexicographically ordered family. Thus, we may assume w.l.o.g. that  $\mu(\mathcal{F}) \leq \frac{1}{2}$ . Secondly, note that the conclusion of the theorem holds trivially if  $C\epsilon \geq 2\mu$ . So we may assume throughout that  $\epsilon < \frac{2\mu}{C}$ .

Provided  $C \geq 2/c_1$ , we have  $\epsilon \leq c_1\mu$ , so either Case (1) or Case (2) of Proposition 4.1 occurs. First suppose that Case (1) occurs. By replacing  $\mathcal{F}$  by a family  $\mathcal{G}$  weakly isomorphic to  $\mathcal{F}$  if necessary, we may assume that

$$(4.1) \quad c_2\mu_1^-(\mathcal{F}) + \frac{1}{2}\epsilon_1^+(\mathcal{F}) \leq \epsilon.$$

Assume w.l.o.g. that the minimum of  $\mu(\mathcal{F}_{\{1\}}^{\{1\}} \Delta \mathcal{H})$  over all families  $\mathcal{H}$  weakly isomorphic to  $\mathcal{L}_{\mu}(\mathcal{F}_{\{1\}}^{\{1\}})$  is attained where  $\mathcal{H} = \mathcal{L}_{\mu}(\mathcal{F}_{\{1\}}^{\{1\}})$ , i.e. where  $\mathcal{H}$  is a lexicographically ordered family with respect to the usual ordering  $2 \leq 3 \leq \dots \leq n$ .

Note that

$$(4.2) \quad \mu(\mathcal{F} \Delta \mathcal{L}_{\mu}) = 2\mu(\mathcal{F} \setminus \mathcal{L}_{\mu}) = \mu(\mathcal{F}_{\{1\}}^{\{1\}} \setminus (\mathcal{L}_{\mu})_{\{1\}}^{\{1\}}) + \mu(\mathcal{F}_{\{1\}}^{\emptyset}).$$

The induction hypothesis, and our assumption above on the families  $\mathcal{H}$ , imply that

$$(4.3) \quad \mu(\mathcal{F}_{\{1\}}^{\{1\}} \setminus (\mathcal{L}_{\mu})_{\{1\}}^{\{1\}}) \leq \frac{1}{2}\mu(\mathcal{F}_{\{1\}}^{\{1\}} \Delta \mathcal{L}_{\mu_1^+}) \leq \frac{1}{2}C\epsilon_1^+.$$

Rearranging (4.1), we have

$$(4.4) \quad \epsilon_1^+ \leq 2(\epsilon - c_2\mu_1^-).$$

Putting together (4.2), (4.3) and (4.4), we obtain

$$\mu(\mathcal{F} \Delta \mathcal{L}_{\mu}) \leq C(\epsilon - c_2\mu_1^-) + \mu_1^- \leq C\epsilon,$$

provided  $C \geq \frac{1}{c_2}$ . This completes the proof in Case (1).

Suppose now that Case (2) occurs. Replacing  $\mathcal{F}$  by a family  $\mathcal{G}$  weakly isomorphic to it, we may assume that

$$(4.5) \quad c_2\mu(\mathcal{F} \setminus \mathcal{S}_{\{1,2\}}) + \frac{1}{4}\epsilon_{1,2}^{++}(\mathcal{F}) \leq \epsilon.$$

Assume w.l.o.g. that the minimum of  $\mu(\mathcal{F}_{\{1,2\}}^{\{1,2\}} \Delta \mathcal{H})$  over all families  $\mathcal{H}$  weakly isomorphic to  $\mathcal{L}_{\mu}(\mathcal{F}_{\{1,2\}}^{\{1,2\}})$  is attained where  $\mathcal{H} = \mathcal{L}_{\mu}(\mathcal{F}_{\{1,2\}}^{\{1,2\}})$ , i.e. where  $\mathcal{H}$  is a lexicographically ordered family with respect to the usual ordering  $3 \leq \dots \leq n$ .

We now have

$$(4.6) \quad \mu(\mathcal{F} \Delta \mathcal{L}_{\mu}) \leq \frac{1}{4}\mu(\mathcal{F}_{\{1,2\}}^{\{1,2\}} \Delta \mathcal{L}_{\mu_{1,2}^{++}(\mathcal{F})}) + 2\mu(\mathcal{F} \setminus \mathcal{S}_{\{1,2\}}).$$

By the induction hypothesis,

$$(4.7) \quad \mu(\mathcal{F}_{\{1,2\}}^{\{1,2\}} \Delta \mathcal{L}_{\mu_{1,2}^{++}(\mathcal{F})}) \leq C\epsilon_{1,2}^{++}.$$

Putting together (4.5), (4.6) and (4.7), we obtain

$$\mu(\mathcal{F} \Delta \mathcal{L}_{\mu}) \leq C(\epsilon - c_2\mu(\mathcal{F} \setminus \mathcal{S}_{\{1,2\}})) + 2\mu(\mathcal{F} \setminus \mathcal{S}_{\{1,2\}}) \leq C\epsilon,$$

where the last inequality holds provided  $C \geq \frac{2}{c_2}$ . This completes the proof.  $\square$

## 5. FRACTIONAL LEXICOGRAPHIC FAMILIES AND THEIR PROPERTIES

A *fractional lexicographic family of order  $n$*  is a function  $\mathcal{F}: \mathcal{P}([n]) \rightarrow \mathbb{D}$ , where  $\mathbb{D} = \{\frac{b}{2^a} : a \in \mathbb{N}, b \in \{0, 1, \dots, 2^a\}\}$  denotes the set of dyadic rationals between 0 and 1. Intuitively, a fractional lexicographic family  $\mathcal{F}: \mathcal{P}([n]) \rightarrow \mathbb{D}$  represents a (non-fractional) family  $\mathcal{G} \subset \mathcal{P}([n+m])$ , for some  $m \in \mathbb{N}$ , such that  $\mathcal{G}_{[n]}^B$  is a lexicographically ordered family of measure  $\mathcal{F}(B)$ , for each  $B \subset [n]$ . Formally, if  $\mathcal{F}: \mathcal{P}([n]) \rightarrow \mathbb{D}$ , we choose any  $m \in \mathbb{N}$  such that  $2^m \mathcal{F}(\mathcal{P}([n])) \subset \mathbb{Z}$ , and associate to  $\mathcal{F}$  the family

$$\mathcal{F}_{\text{ass}} \subset \mathcal{P}([n+m])$$

such that  $(\mathcal{F}_{\text{ass}})_{[n]}^B$  is the lexicographically ordered family of measure  $\mathcal{F}(B)$ , for each  $B \subset [n]$ . If  $\mathcal{F}$  is a fractional lexicographic family, then by a slight abuse of notation we define  $\mu(\mathcal{F})$ ,  $I[\mathcal{F}]$ ,  $\mu_i^+(\mathcal{F})$  and  $\mu_i^-(\mathcal{F})$  (for  $i \in [n]$ ) to be the corresponding quantities for the associated family  $\mathcal{F}_{\text{ass}} \subset \mathcal{P}([n+m])$ ; it is easy to see that these are independent of the choice of  $m$ , provided we define  $\text{Inf}_i[\mathcal{F}_{\text{ass}}] = 0$  for all  $i > n+m$ .

The usefulness of fractional lexicographic families comes from the fact that inductive arguments enable us to reduce statements about general families to statements about fractional lexicographic families of order  $n$ , for small  $n$ . Specifically, we need a thorough analysis of the case  $n = 1$  and the case  $n = 2$ . These statements encapsulate the idea that families of small total influence can only have their measure split between two disjoint codimension-1 subcubes (or between four disjoint codimension-2 subcubes) in certain ways. The proofs of the statements are technical and the reader is advised (at least at first reading) to read the statements of the lemmas without going into their proofs.

**5.1. Properties of fractional lexicographic families of order 1.** If  $0 \leq \mu^-, \mu^+ \leq 1$ , we denote by  $\mathcal{L}_{\mu^-, \mu^+}$  the fractional lexicographic family  $\mathcal{L}: \{\emptyset, \{1\}\} \rightarrow [0, 1]$  of order 1, with  $\mathcal{L}(\{1\}) = \mu^+$  and  $\mathcal{L}(\emptyset) = \mu^-$ .

Let  $\mu = 2^{-j} + r$ , where  $j \geq 2$  and  $0 < r \leq 2^{-j}$ . Observe that

$$(5.1) \quad \mu_i^-(\mathcal{L}_\mu) \begin{cases} = 0 & \text{if } i \leq j-1, \\ = 2r & \text{if } i = j, \\ \geq \frac{1}{2}\mu & \text{if } i \geq j+1. \end{cases}$$

The next lemma says roughly that if a fractional lexicographic family  $\mathcal{L} = \mathcal{L}_{\mu^-, \mu^+}$  of order 1 has  $0 < \mu^- \leq r$ , then  $I[\mathcal{L}_{\mu^-, \mu^+}]$  is somewhat large.

**Lemma 5.1.** *Let  $j \geq 2$ , let  $0 < r \leq 2^{-j}$ , let  $\mu = 2^{-j} + r$ , and let  $0 \leq \mu^- \leq \mu^+ \leq 1$  with  $\frac{\mu^- + \mu^+}{2} = \mu$ . If  $\mu^- \leq r$ , then  $I[\mathcal{L}_{\mu^-, \mu^+}] \geq I[\mathcal{L}_\mu] + 2\mu^-$ .*

*If instead,  $3r \leq \mu^- \leq \frac{1}{2}\mu$ , then  $I[\mathcal{L}_{\mu^-, \mu^+}] \geq I[\mathcal{L}_\mu] + \frac{2}{3}\mu^-$ .*

In order to prove the lemma, we need the following preparatory claim.

**Claim 5.2.** *Let  $\mu = \frac{\mu^- + \mu^+}{2} = \frac{1}{4} + r$ , where  $0 < r \leq 1/4$  and  $0 \leq \mu^-, \mu^+ \leq 1$ . Suppose that  $\mu^- \leq r$ . Then*

$$I[\mathcal{L}_{\mu^-, \mu^+}] \geq I[\mathcal{L}_\mu] + 2\mu^-.$$

*Proof.* Write  $\mathcal{L} = \mathcal{L}_{\mu^-, \mu^+}$ . Then we may view  $\mathcal{L}$  as a fractional lexicographical family on  $\mathcal{P}([2])$  such that  $\mathcal{L}(\{1, 2\}) = 1$ ,  $\mathcal{L}(\emptyset) = \emptyset$ ,  $\mathcal{L}(\{2\}) = 2\mu^-$ , and  $\mathcal{L}(\{1\}) =$

$2\mu^+ - 1$ . Using Lemma 3.1 (and writing  $\mathcal{L}$  in place of  $\mathcal{L}_{\text{ass}}$  in the first line of (5.2) below), we have

$$\begin{aligned}
 (5.2) \quad I[\mathcal{L}] &= \frac{1}{4} \sum_{B \subset \{1,2\}} I[\mathcal{L}_{\{1,2\}}^B] + \frac{1}{2} \left( \mu(\mathcal{L}_{\{1,2\}}^{\{1,2\}}) - \mu(\mathcal{L}_{\{1,2\}}^{\{1\}}) \right) \\
 &\quad + \frac{1}{2} \left( \mu(\mathcal{L}_{\{1,2\}}^{\{1,2\}}) - \mu(\mathcal{L}_{\{1,2\}}^{\{2\}}) \right) \\
 &\quad + \frac{1}{2} \left( \mu(\mathcal{L}_{\{1,2\}}^{\{1\}}) - \mu(\mathcal{L}_{\{1,2\}}^{\emptyset}) \right) + \frac{1}{2} \left( \mu(\mathcal{L}_{\{1,2\}}^{\{2\}}) - \mu(\mathcal{L}_{\{1,2\}}^{\emptyset}) \right) \\
 &= \frac{1}{4} (I[\mathcal{L}_{2\mu^-}] + I[\mathcal{L}_{2\mu^+ - 1}]) + \mathcal{L}(\{1,2\}) - \mathcal{L}(\emptyset).
 \end{aligned}$$

Similarly,  $\mathcal{M} := \mathcal{L}_\mu$  may be viewed as a lexicographical family on  $\mathcal{P}([2])$  with  $\mathcal{M}(\emptyset) = \emptyset$ ,  $\mathcal{M}(\{2\}) = \emptyset$ ,  $\mathcal{M}(\{1,2\}) = 1$ , and  $\mathcal{M}(\{1\}) = 4\mu - 1$ . So as in (5.2), we have

$$(5.3) \quad I[\mathcal{M}] = \frac{1}{4} I[\mathcal{L}_{4\mu-1}] + \mathcal{L}(\{1,2\}) - \mathcal{L}(\emptyset).$$

Putting (5.2) and (5.3) together, we have

$$(5.4) \quad I[\mathcal{L}] - I[\mathcal{M}] = \frac{1}{4} (I[\mathcal{L}_{2\mu^-}] + I[\mathcal{L}_{2\mu^+ - 1}] - I[\mathcal{L}_{4\mu-1}]).$$

We now consider the families  $\mathcal{F}_1 := \mathcal{L}_{\frac{4\mu-1}{2}}$  and  $\mathcal{F}_2 := \mathcal{L}_{2\mu^-, 2\mu^+ - 1}$ . The isoperimetric inequality implies that

$$(5.5) \quad I[\mathcal{F}_2] \geq I[\mathcal{F}_1].$$

Let us compute the influences of  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . We have

$$(5.6) \quad I[\mathcal{F}_1] = \frac{1}{2} I[\mathcal{L}_{4\mu-1}] + 4\mu - 1$$

and

$$(5.7) \quad I[\mathcal{F}_2] = \frac{1}{2} I[\mathcal{L}_{2\mu^-}] + \frac{1}{2} I[\mathcal{L}_{2\mu^+ - 1}] + 2\mu^+ - 1 - 2\mu^-.$$

(Here, we used the fact that  $2\mu^+ - 1 = 4\mu - 2\mu^- - 1 = 4r - 2\mu^- \geq 2\mu^-$ .)

Combining (5.4)-(5.7) yields

$$\begin{aligned}
 I[\mathcal{L}] - I[\mathcal{M}] &= \frac{1}{4} (I[\mathcal{L}_{2\mu^-}] + I[\mathcal{L}_{2\mu^+ - 1}] - I[\mathcal{L}_{4\mu-1}]) \\
 &= \frac{1}{2} (I[\mathcal{F}_2] - (2\mu^+ - 1 - 2\mu^-)) \\
 &\quad - \frac{1}{2} (I[\mathcal{F}_1] - (2\mu^+ + 2\mu^- - 1)) \\
 &\geq 2\mu^-,
 \end{aligned}$$

as required.  $\square$

We can now prove Lemma 5.1.

*Proof of Lemma 5.1.* We prove the first statement by induction on  $j$ . Suppose that  $\mu^- \leq r$ . Since  $\mu^+ = 2\mu - \mu^- > (\frac{1}{2})^{j-1}$ , we have  $j \geq 2$ . Claim 5.2 implies the base case  $j = 2$ . Let  $j \geq 3$ , and assume the statement holds for smaller values of  $j$ . Since  $\mu \leq 1/8$ ,  $(\mathcal{L}_{\mu^-, \mu^+})_{\text{ass}}$  is contained in the dictatorship  $\mathcal{D}_2$ . By Lemma 3.1, we have

$$\begin{aligned}
 (5.8) \quad I[\mathcal{L}_{\mu^-, \mu^+}] &= \frac{1}{2} I[(\mathcal{L}_{\mu^-, \mu^+})_{\{2\}}^{\{2\}}] + \frac{1}{2} I[(\mathcal{L}_{\mu^-, \mu^+})_{\{2\}}^{\{\emptyset\}}] + \text{Inf}_2[\mathcal{L}_{\mu^-, \mu^+}] \\
 &= \frac{1}{2} I[\mathcal{L}_{2\mu^-, 2\mu^+}] + 2\mu.
 \end{aligned}$$

Similarly,

$$(5.9) \quad I[\mathcal{L}_\mu] = \frac{1}{2} I[\mathcal{L}_{2\mu}] + 2\mu.$$

The induction hypothesis implies that

$$(5.10) \quad I[\mathcal{L}_{2\mu^-, 2\mu^+}] \geq I[\mathcal{L}_{2\mu}] + 4\mu^-.$$

Combining (5.8), (5.9), and (5.10), we obtain  $I[\mathcal{L}_{\mu^-, \mu^+}] \geq I[\mathcal{L}_\mu] + 2\mu^-$ , as required.

Now suppose that  $3r \leq \mu^- \leq \frac{1}{2}\mu$ . We proceed again by induction on  $j$ . First suppose  $j = 1$ . Note that  $I[\mathcal{L}_{1-\mu^+, 1-\mu^-}] = I[\mathcal{L}_{\mu^-, \mu^+}]$ , and that  $\mu(\mathcal{L}_{1-\mu^+, 1-\mu^-}) = \frac{1}{2} - r = \frac{1}{4} + (\frac{1}{4} - r)$ . We also have

$$\mu^+ = 2\mu - \mu^- \geq \frac{3}{4} + \frac{3}{2}r.$$

Hence,

$$1 - \mu^+ \leq \frac{1}{4} - \frac{3}{2}r \leq \frac{1}{4} - r.$$

Claim 5.2 implies that

$$\begin{aligned} I[\mathcal{L}_{\mu^-, \mu^+}] &= I[\mathcal{L}_{1-\mu^+, 1-\mu^-}] \\ &\geq I[\mathcal{L}_{1-\mu}] + 2(1 - \mu^+) \\ &= I[\mathcal{L}_\mu] + 2(1 - 2\mu + \mu^-) \\ &= I[\mathcal{L}_\mu] + 2(\mu^- - 2r) \\ &\geq I[\mathcal{L}_\mu] + \frac{2}{3}\mu^-, \end{aligned}$$

as desired. The inductive step is almost exactly the same as in the previous case, relying on the fact that  $(\mathcal{L}_{\mu^-, \mu^+})_{\text{ass}}$  is contained in the dictatorship  $\mathcal{D}_2$ .  $\square$

**5.2. Properties of fractional lexicographic families of order 2.** By (5.1), if  $\mu \leq 1/2$  then there is a unique value of  $i$  ( $i = j$ , say) for which  $0 < \mu_i^-(\mathcal{L}_\mu) < \frac{1}{2}\mu$ . Let  $\mathcal{L}: \mathcal{P}([2]) \rightarrow [0, 1]$  be a fractional lexicographic family of order 2. The following lemma says that if both  $\mu_1^-(\mathcal{L})$  and  $\mu_2^-(\mathcal{L})$  are similar to  $\mu_j^-(\mathcal{L}_\mu)$ , then  $\mathcal{L}$  has ‘somewhat large’ total influence.

**Lemma 5.3.** *There exists an absolute constant  $c > 0$  such that the following holds. Let  $\mu = 2^{-i} + r$ , where  $i \in \mathbb{N}$  and  $r \leq 2^{-i}$ . Let  $\mathcal{L}: \mathcal{P}([2]) \rightarrow [0, 1]$  be a fractional lexicographic family of order 2 and measure  $\mu$ . Suppose that  $r \leq \mu_1^-(\mathcal{L}) \leq 3r$ , that  $r \leq \mu_2^-(\mathcal{L}) \leq 3r$ , and that  $r \leq c\mu$ . Then  $I[\mathcal{L}] \geq I[\mathcal{L}_\mu] + r/2$ .*

(In fact, we may take  $c = 1/6$ .)

*Proof.* Suppose w.l.o.g. that  $\mathcal{L}(\{2\}) \leq \mathcal{L}(\{1\})$ . We split into two cases:  $\mathcal{L}(\{2\}) \geq \frac{r}{2}$ , and  $\mathcal{L}(\{2\}) \leq \frac{r}{2}$ .

First suppose that  $\mathcal{L}(\{2\}) \geq \frac{r}{2}$ . Note that, by hypothesis,

$$\frac{1}{2}\mathcal{L}(\emptyset) + \frac{1}{2}\mathcal{L}(\{2\}) = \mu_1^-(\mathcal{L}) \leq 3r, \quad \frac{1}{2}\mathcal{L}(\emptyset) + \frac{1}{2}\mathcal{L}(\{1\}) = \mu_2^-(\mathcal{L}) \leq 3r,$$

so

$$\begin{aligned} \mathcal{L}(\{1, 2\}) &= 4\mu - \mathcal{L}(\emptyset) - \mathcal{L}(\{1\}) - \mathcal{L}(\{2\}) \\ &\geq 4\mu - 2\mathcal{L}(\emptyset) - \mathcal{L}(\{1\}) - \mathcal{L}(\{2\}) \\ &\geq 4\mu - 12r \\ &\geq 4r, \end{aligned}$$

provided  $c \leq 1/4$ . By Lemma 3.1, we have

$$(5.11) \quad I[\mathcal{L}] \geq \frac{1}{2}I[\mathcal{L}_{\{2\}}^{\{2\}}] + \frac{1}{2}I[\mathcal{L}_{\{2\}}^{\emptyset}] + \mu_2^+(\mathcal{L}) - \mu_2^-(\mathcal{L}).$$

Let  $\mathcal{L}'$  be the fractional lexicographic family of order 2, such that

$$\mathcal{L}'(\emptyset) = 0, \mathcal{L}'(\{1\}) = 2\mu_2^-(\mathcal{L}), \mathcal{L}'(\{2\}) = \mathcal{L}(\{2\}), \mathcal{L}'(\{1, 2\}) = \mathcal{L}(\{1, 2\}).$$

Note that  $\mu_2^-(\mathcal{L}) \leq 1/2$  provided  $c \leq 1/6$ , that  $\mathcal{L}'(\{2\}) = \mathcal{L}(\{2\}) \leq \mathcal{L}(\{1\}) \leq 2\mu_2^-(\mathcal{L}) = \mathcal{L}'(\{1\})$ , and that  $\mathcal{L}'(\{1, 2\}) \geq \mathcal{L}'(\{1\})$  provided  $c \leq 2/9$ .

By Lemma 3.1, we have

$$(5.12) \quad I[\mathcal{L}'] = \frac{1}{2}I[\mathcal{L}_{\{2\}}^{\{2\}}] + \frac{1}{2}I[\mathcal{L}_{\mu_2^-}] + \mu_2^+(\mathcal{L}) - \mu_2^-(\mathcal{L}).$$

By the isoperimetric inequality, (5.11) and (5.12), we have

$$(5.13) \quad I[\mathcal{L}] \geq I[\mathcal{L}'].$$

Also by Lemma 3.1, we have

$$(5.14) \quad I[\mathcal{L}'] = \frac{1}{4}I[\mathcal{L}_{\mathcal{L}(\{1,2\})}] + \frac{1}{4}I[\mathcal{L}_{\mathcal{L}'(\{1\})}] + \frac{1}{4}I[\mathcal{L}_{\mathcal{L}'(\{2\})}] + \mathcal{L}(\{1, 2\}).$$

Let  $\mathcal{L}''$  be the fractional lexicographic family of order 2, such that

$$\mathcal{L}''(\emptyset) = 0, \mathcal{L}''(\{2\}) = 0, \mathcal{L}''(\{1\}) = \mathcal{L}'(\{1\}) + \mathcal{L}'(\{2\}), \mathcal{L}''(\{1, 2\}) = \mathcal{L}(\{1, 2\}).$$

Note that  $\mathcal{L}'(\{1\}) + \mathcal{L}'(\{2\}) \leq \mathcal{L}(\{1, 2\})$  provided  $c \leq 1/6$ . By Lemma 3.1, we have

$$(5.15) \quad I[\mathcal{L}''] = \frac{1}{4}I[\mathcal{L}_{\mathcal{L}(\{1,2\})}] + \frac{1}{4}I[\mathcal{L}_{\mathcal{L}'(\{1\})+\mathcal{L}'(\{2\})}] + \mathcal{L}(\{1, 2\}).$$

The isoperimetric inequality implies that

$$\begin{aligned} I[\mathcal{L}_{\mathcal{L}'(\{2\}), \mathcal{L}'(\{1\})}] &\geq I\left[\mathcal{L}_{\frac{\mathcal{L}'(\{2\})+\mathcal{L}'(\{1\})}{2}}\right] \\ &= I[\mathcal{L}_{0, \mathcal{L}'(\{1\})+\mathcal{L}'(\{2\})}] \\ &= \frac{1}{2}I[\mathcal{L}_{\mathcal{L}'(\{1\})+\mathcal{L}'(\{2\})}] + \mathcal{L}'(\{1\}) + \mathcal{L}'(\{2\}). \end{aligned}$$

Applying Lemma 3.1, we obtain

$$\begin{aligned} &\frac{1}{2}I[\mathcal{L}_{\mathcal{L}'(\{1\})}] + \frac{1}{2}I[\mathcal{L}_{\mathcal{L}'(\{2\})}] + \mathcal{L}'(\{1\}) - \mathcal{L}'(\{2\}) \\ &= I[\mathcal{L}_{\mathcal{L}'(\{2\}), \mathcal{L}'(\{1\})}] \\ &\geq \frac{1}{2}I[\mathcal{L}_{\mathcal{L}'(\{1\})+\mathcal{L}'(\{2\})}] + \mathcal{L}'(\{1\}) + \mathcal{L}'(\{2\}), \end{aligned}$$

so rearranging,

$$(5.16) \quad \frac{1}{2}I[\mathcal{L}_{\mathcal{L}'(\{1\})}] + \frac{1}{2}I[\mathcal{L}_{\mathcal{L}'(\{2\})}] \geq \frac{1}{2}I[\mathcal{L}_{\mathcal{L}'(\{1\})+\mathcal{L}'(\{2\})}] + 2\mathcal{L}'(\{2\}).$$

Putting everything together, we have

$$\begin{aligned} I[\mathcal{L}] &\geq I[\mathcal{L}'] \\ &\geq \frac{1}{4}I[\mathcal{L}_{\mathcal{L}(\{1,2\})}] + \mathcal{L}(\{1, 2\}) + \frac{1}{4}I[\mathcal{L}_{\mathcal{L}'(\{1\})+\mathcal{L}'(\{2\})}] + \mathcal{L}'(\{2\}) \\ &= I[\mathcal{L}''] + \mathcal{L}(\{2\}) \\ &\geq I[\mathcal{L}_\mu] + \mathcal{L}(\{2\}) \\ &\geq I[\mathcal{L}_\mu] + r/2, \end{aligned}$$

as required.

Suppose now that  $\mathcal{L}(\{2\}) \leq \frac{r}{2}$ . Since  $\mu_1^-(\mathcal{L}) \geq r$ , we have  $\mathcal{L}(\emptyset) \geq \frac{3r}{2}$ . Let  $\mathcal{L}'$  be the fractional lexicographic family of order 2 such that

$$\begin{aligned} \mathcal{L}'(\emptyset) &= 0, \mathcal{L}'(\{2\}) = \mathcal{L}(\{2\}), \\ \mathcal{L}'(\{1\}) &= \mathcal{L}(\emptyset) + \mathcal{L}(\{1\}), \mathcal{L}'(\{1, 2\}) = \mathcal{L}(\{1, 2\}). \end{aligned}$$

Note that  $\mathcal{L}(\emptyset) + \mathcal{L}(\{1\}) \leq \mathcal{L}(\{1, 2\})$ , provided  $c \leq 2/9$ . By Lemma 3.1, we have

$$\begin{aligned} I[\mathcal{L}] &= \frac{1}{4} (I[\mathcal{L}_{\mathcal{L}(\emptyset)}] + I[\mathcal{L}_{\mathcal{L}(\{1\})}] + I[\mathcal{L}_{\mathcal{L}(\{2\})}] + I[\mathcal{L}_{\mathcal{L}(\{1,2\})}]) \\ &\quad + \mathcal{L}(\{1, 2\}) - \frac{1}{2}\mathcal{L}(\{1\}) - \mathcal{L}(\{2\}) + \frac{1}{2}\mathcal{L}(\emptyset) + \frac{1}{2}|\mathcal{L}(\emptyset) - \mathcal{L}(\{1\})|, \end{aligned}$$

and

$$I[\mathcal{L}'] = \frac{1}{4} (I[\mathcal{L}_{\mathcal{L}(\emptyset)+\mathcal{L}(\{1\})}] + I[\mathcal{L}_{\mathcal{L}(\{2\})}] + I[\mathcal{L}_{\mathcal{L}(\{1,2\})}]) + \mathcal{L}(\{1, 2\}),$$

and therefore

$$\begin{aligned} I[\mathcal{L}] - I[\mathcal{L}'] &= \frac{1}{4} (I[\mathcal{L}_{\mathcal{L}(\emptyset)}] + I[\mathcal{L}_{\mathcal{L}(\{1\})}] - I[\mathcal{L}_{\mathcal{L}(\emptyset)+\mathcal{L}(\{1\})}]) \\ &\quad + \frac{1}{2}(\mathcal{L}(\emptyset) - \mathcal{L}(\{1\})) + \frac{1}{2}|\mathcal{L}(\emptyset) - \mathcal{L}(\{1\})| - \mathcal{L}(\{2\}) \\ &= \frac{1}{4} (I[\mathcal{L}_{\mathcal{L}(\emptyset)}] + I[\mathcal{L}_{\mathcal{L}(\{1\})}] - I[\mathcal{L}_{\mathcal{L}(\emptyset)+\mathcal{L}(\{1\})}]) \\ (5.17) \quad &\quad + \min\{\mathcal{L}(\emptyset) - \mathcal{L}(\{1\}), 0\} - \mathcal{L}(\{2\}). \end{aligned}$$

For all  $\mu^-, \mu^+ \geq 0$  such that  $\mu^- + \mu^+ \leq 1$ , we have

$$\frac{1}{2}I[\mathcal{L}_{\mu^-}] + \frac{1}{2}I[\mathcal{L}_{\mu^+}] + |\mu^+ - \mu^-| = I[\mathcal{L}_{\mu^+, \mu^-}] \geq \frac{1}{2}I[\mathcal{L}_{\mu^+ + \mu^-}] + \mu^+ + \mu^-,$$

using Lemma 3.1 and the isoperimetric inequality, so

$$I[\mathcal{L}_{\mu^-}] + I[\mathcal{L}_{\mu^+}] - I[\mathcal{L}_{\mu^+ + \mu^-}] \geq 4 \min\{\mu^+, \mu^-\}.$$

Applying this with  $\mu^- = \mathcal{L}(\emptyset)$  and  $\mu^+ = \mathcal{L}(\{1\})$  gives

$$I[\mathcal{L}_{\mathcal{L}(\emptyset)}] + I[\mathcal{L}_{\mathcal{L}(\{1\})}] - I[\mathcal{L}_{\mathcal{L}(\emptyset)+\mathcal{L}(\{1\})}] \geq 4 \min\{\mathcal{L}(\emptyset), \mathcal{L}(\{1\})\}.$$

Combining this with (5.17) yields

$$\begin{aligned} I[\mathcal{L}] - I[\mathcal{L}'] &\geq \min\{\mathcal{L}(\emptyset), \mathcal{L}(\{1\})\} + \min\{\mathcal{L}(\emptyset) - \mathcal{L}(\{1\}), 0\} - \mathcal{L}(\{2\}) \\ &= \mathcal{L}(\emptyset) - \mathcal{L}(\{2\}) \\ &\geq \frac{3r}{2} - \frac{r}{2} \\ &= r. \end{aligned}$$

The isoperimetric inequality now implies that

$$I[\mathcal{L}] \geq I[\mathcal{L}'] + r \geq I[\mathcal{L}_{\mu}] + r,$$

completing the proof.  $\square$

## 6. TWO ‘BOOTSTRAPPING’ LEMMAS

In this section, we prove two ‘bootstrapping’ lemmas which say, roughly speaking, that if  $\mathcal{F}$  is ‘somewhat’ close to being contained in a subcube of codimension 1 or 2, then it is ‘very’ close to being contained in that subcube. In what follows, we write  $\mu = 2^{-j} + r$ , where  $j \geq 2$  and  $0 < r \leq 2^{-j}$ , we let  $\epsilon > 0$ , and we let  $\mathcal{F} \subset \mathcal{P}([n])$  be a family with measure  $\mu(\mathcal{F}) = \mu$ , and with  $I[\mathcal{F}] = I[\mathcal{L}_{\mu}] + \epsilon$ .

Recall that our goal is to prove Proposition 4.1. First, we deal with the case where  $r$  is ‘large’. In this case, our aim is to show that  $\min\{2\mu_i^- + \frac{1}{2}\epsilon_i^+, 2\mu_i^+ + \frac{1}{2}\epsilon_i^-\} \leq \epsilon$  for some  $i \in [n]$ . The following ‘bootstrapping’ lemma says that this inequality holds provided only that  $\mu_i^-(\mathcal{F}) \leq r$ .

**Lemma 6.1.** *Let  $0 \leq \mu \leq \frac{1}{2}$  and write  $\mu = 2^{-j} + r$ , where  $j \geq 2$  and  $r \leq 2^{-j}$ . Let  $\mathcal{F} \subset \mathcal{P}([n])$  be a family with measure  $\mu(\mathcal{F}) = \mu$ , and with  $I[\mathcal{F}] = I[\mathcal{L}_{\mu}] + \epsilon$ . If  $\mu_i^- := \mu_i^-(\mathcal{F}) \leq r$  for some  $i \in [n]$ , then  $2\mu_i^- + \frac{1}{2}\epsilon_i^+ \leq \epsilon$ .*

*Proof.* Using Lemma 3.1, the isoperimetric inequality and Lemma 5.1, we have

$$\begin{aligned}
I[\mathcal{F}] &= \frac{1}{2}I[\mathcal{F}_{\{i\}}^{\{i\}}] + \frac{1}{2}I[\mathcal{F}_{\{i\}}^{\emptyset}] + \text{Inf}_i[\mathcal{F}] \\
&\geq \frac{1}{2}I[\mathcal{L}_{\mu_i^+}] + \frac{1}{2}\epsilon_i^+ + \frac{1}{2}I[\mathcal{L}_{\mu_i^-}] + \mu_i^+ - \mu_i^- \\
&= I[\mathcal{L}_{\mu_i^-, \mu_i^+}] + \frac{1}{2}\epsilon_i^+ \\
&\geq I[\mathcal{L}_\mu] + 2\mu_i^- + \frac{1}{2}\epsilon_i^+.
\end{aligned}$$

Rearranging yields

$$2\mu_i^- + \frac{1}{2}\epsilon_i^+ \leq I[\mathcal{F}] - I[\mathcal{L}_\mu] = \epsilon,$$

proving the lemma.  $\square$

We now prove a bootstrapping lemma suitable for the case where  $r$  is ‘small’. Here, our final goal is to show that there exists a family  $\mathcal{G}$  weakly isomorphic to  $\mathcal{F}$  such that either  $c\mu_1^-(\mathcal{G}) + \frac{1}{2}\epsilon_1^+(\mathcal{G}) \leq \epsilon$ , or else  $c\mu(\mathcal{G} \setminus \mathcal{S}_{\{1,2\}}) + \frac{1}{4}\epsilon_{1,2}^{++}(\mathcal{G}) \leq \epsilon$ . We show that one of these inequalities holds provided  $\mu_1^-(\mathcal{G}) \leq \mu_2^-(\mathcal{G}) \leq c\mu$ , if  $c$  is a sufficiently small positive constant.

**Lemma 6.2.** *Let  $\epsilon > 0$ , let  $0 < \mu \leq \frac{1}{2}$ , and write  $\mu = 2^{-j} + r$ , where  $j \geq 2$  and  $0 < r \leq 2^{-j}$ . Let  $\mathcal{F} \subset \mathcal{P}([n])$  be a family with measure  $\mu(\mathcal{F}) = \mu$ , and with  $I[\mathcal{F}] = I[\mathcal{L}_\mu] + \epsilon$ . Suppose that  $\mu_1^-(\mathcal{F}) \leq \mu_2^-(\mathcal{F}) \leq \frac{1}{6}\mu$ . Then either  $\frac{2}{3}\mu_2^-(\mathcal{F}) + \frac{1}{2}\epsilon_2^+(\mathcal{F}) \leq \epsilon$ , or  $2\mu_1^-(\mathcal{F}) + \frac{1}{2}\epsilon_1^+(\mathcal{F}) \leq \epsilon$ , or  $\frac{1}{6}\mu(\mathcal{F} \setminus \mathcal{S}_{\{1,2\}}) + \frac{1}{4}\epsilon_{1,2}^{++}(\mathcal{F}) \leq \epsilon$ .*

*Proof.* The case where  $\mu_1^- \leq r$  is covered by Lemma 6.1, and the case where  $\mu_2^- \geq 3r$  can be covered similarly by using the second part of Lemma 5.1 instead of its first part. So we may assume that  $r \leq \mu_1^- \leq \mu_2^- \leq 3r$ .

Let  $\mathcal{L}$  be the fractional lexicographic family of order 2, with

$$\mathcal{L}(\{1, 2\}) = \mu_{1,2}^{++}, \quad \mathcal{L}(\{1\}) = \mu_{1,2}^{+-}, \quad \mathcal{L}(\{2\}) = \mu_{1,2}^{-+}, \quad \mathcal{L}(\{\emptyset\}) = \mu_{1,2}^{--}.$$

Using Lemma 3.1, the isoperimetric inequality and Lemma 5.3, we obtain

$$I[\mathcal{F}] \geq I[\mathcal{L}] + \frac{1}{4}\epsilon_{1,2}^{++} \geq I[\mathcal{L}_\mu] + \frac{1}{4}\epsilon_{1,2}^{++} + \frac{1}{2}r,$$

and therefore

$$\epsilon \geq \frac{1}{4}\epsilon_{1,2}^{++} + \frac{1}{2}r \geq \frac{1}{4}\epsilon_{1,2}^{++} + \frac{1}{12}(\mu_1^- + \mu_2^-) \geq \frac{1}{4}\epsilon_{1,2}^{++} + \frac{1}{6}\mu(\mathcal{F} \setminus \mathcal{S}_{\{1,2\}}),$$

proving the lemma.  $\square$

## 7. $\mathcal{F}$ IS ESSENTIALLY CONTAINED IN A CODIMENSION-1 SUBCUBE ( $\mu$ LARGE)

In this section we essentially complete the proof of Proposition 4.1 in the case where  $\mu(\mathcal{F}) = \frac{1}{4} + r$ , for  $r \leq \frac{1}{4}$  bounded away from 0 (i.e.  $r \geq c_1$  for some absolute constant  $c_1 > 0$ ). In this case, by Lemma 6.1 it will suffice to prove the following.

**Proposition 7.1.** *For each  $c_1 > 0$  there exists  $c_2 = c_2(c_1) > 0$  such that the following holds. If  $\mathcal{F} \subset \mathcal{P}([n])$  is a family with  $\frac{1}{4} + c_1 \leq \mu := \mu(\mathcal{F}) \leq \frac{1}{2}$  and  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_2$ , then there exists a family  $\mathcal{G}$  weakly isomorphic to  $\mathcal{F}$  such that  $\mu_1^-(\mathcal{G}) \leq c_1$ .*

Equivalently, the conclusion of Proposition 7.1 can be restated by saying that there exists a coordinate  $i \in [n]$  such that  $\min\{\mu_i^-(\mathcal{F}), \mu_i^+(\mathcal{F})\} \leq c_1$ .



Throughout this section,  $\mathcal{F} \subset \mathcal{P}([n])$  will be a family with  $\frac{1}{4} + c_1 \leq \mu := \mu(\mathcal{F}) \leq \frac{1}{2}$ , and with  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_2$ , where  $c_2$  will be sufficiently small in terms of  $c_1$ . We assume without loss of generality that  $\mu_i^- \leq \mu_i^+$  for each  $i \in [n]$ , and that

$$\inf_n[\mathcal{F}] \leq \inf_{n-1}[\mathcal{F}] \leq \cdots \leq \inf_1[\mathcal{F}].$$

We also assume that  $c_1 = 2^{-k}$  for some  $k \in \mathbb{N}$ .

The proof of Proposition 7.1 consists of the following six steps, similarly to [23].

- (1) We show that there is a ‘gap’ between ‘good’ families that satisfy the proposition, and ‘bad’ families which would furnish a counterexample to it. More precisely, we show that if  $\mu_i^-(\mathcal{F}) > c_1$  for each  $i \in [n]$  and if  $\mathcal{F}_2 \subset \mathcal{P}([n])$  is a family with  $\mu(\mathcal{F}_2) = \mu(\mathcal{F})$  and with  $I[\mathcal{F}_2] \leq I[\mathcal{F}]$ , which satisfies  $\mu_i^-(\mathcal{F}_2) \leq c_1$  for some  $i \in [n]$ , then  $\mu(\mathcal{F} \Delta \mathcal{F}_2) > \frac{c_1}{2}$ .
- (2) We reduce the proposition to the case where  $\mathcal{F}$  is increasing.
- (3) We prove the proposition in the case where  $\mathcal{F}$  depends on a constant  $O_{c_2}(1)$  number of coordinates.
- (4) In the other case, where  $n$  is large, we show that the ‘ $n$ -stable’ family

$$\tilde{\mathcal{F}} := \mathcal{S}_{n,n-1}(\mathcal{S}_{n,n-2} \cdots (\mathcal{S}_{n,1}(\mathcal{F})))$$

satisfies  $\mu(\tilde{\mathcal{F}}) = \mu(\mathcal{F})$ ,  $I[\tilde{\mathcal{F}}] \leq I[\mathcal{F}]$ , and  $\mu(\tilde{\mathcal{F}} \Delta \mathcal{F}) \leq \frac{c_1}{2}$ . This reduces us to the case where  $\mathcal{F}$  is an increasing,  $n$ -stable family.

- (5) In the case where  $\mathcal{F}$  is  $n$ -stable and  $|\mathcal{I}_n(\mathcal{F})| \geq 2$  (say  $A \neq B \in \mathcal{I}_n(\mathcal{F})$ ), we show that if both  $\mathcal{F}_1 := (\mathcal{F} \setminus \{B\}) \cup \{A \setminus \{n\}\}$  and  $\mathcal{F}_2 := (\mathcal{F} \setminus \{A\}) \cup \{B \setminus \{n\}\}$  are good, then  $\mathcal{F}$  is also good. (Note that  $|\mathcal{I}_n(\mathcal{F}_1)| < |\mathcal{I}_n(\mathcal{F})|$ , and that  $|\mathcal{I}_n(\mathcal{F}_2)| < |\mathcal{I}_n(\mathcal{F})|$ .)
- (6) Step (5) reduces us to the case where  $|\mathcal{I}_n(\mathcal{F})| \leq 1$ , i.e. the family  $\mathcal{F}$  is very evenly balanced in direction  $n$ ; we can then complete the proof by induction on  $n$ .

**7.1. Gap between good families and bad families.** If  $0 \leq s \leq 1$  and  $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$ , we say  $\mathcal{G}$  is an  $s$ -small modification of  $\mathcal{F}$  if  $\mu(\mathcal{G}) = \mu(\mathcal{F})$ ,  $I[\mathcal{G}] \leq I[\mathcal{F}]$  and  $\mu(\mathcal{F} \Delta \mathcal{G}) \leq s$ . If  $\mathcal{F} \subset \mathcal{P}([n])$  and  $c_1, c_2 > 0$ , we say that  $\mathcal{F}$  is *bad* (with respect to  $(c_1, c_2)$ ) if it is a counterexample to Proposition 7.1, and *good* (with respect to  $(c_1, c_2)$ ) otherwise.

**Lemma 7.2.** *Let  $\mathcal{F} \subset \mathcal{P}([n])$  such that  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_2$ . Let  $\mathcal{G}$  be a  $\frac{c_1}{2}$ -small modification of  $\mathcal{F}$ , and suppose that  $c_2 \leq c_1$ . If  $\mathcal{G}$  is good, then so is  $\mathcal{F}$ .*

*Proof.* Suppose for a contradiction that  $\mathcal{F}$  is bad and  $\mathcal{G}$  is good. Then by assumption,

$$I[\mathcal{G}] \leq I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_2.$$

Since  $\mathcal{G}$  is good, we either have  $\mu_i^-(\mathcal{G}) \leq c_1$  or  $\mu_i^+(\mathcal{G}) \leq c_1$  for some  $i \in [n]$ . By Lemma 6.1, this implies that we either have

$$\mu_i^-(\mathcal{G}) \leq \frac{1}{2}c_2 \leq \frac{1}{2}c_1$$

or

$$\mu_i^+(\mathcal{G}) \leq \frac{1}{2}c_2 \leq \frac{1}{2}c_1,$$

since  $c_2 \leq c_1$ . Since  $\mu(\mathcal{G} \Delta \mathcal{F}) \leq \frac{c_1}{2}$ , we have  $\mu(\mathcal{F} \setminus \mathcal{G}) \leq \frac{c_1}{4}$ , and therefore  $\mu_i^-(\mathcal{F}) \leq \mu_i^-(\mathcal{G}) + 2\mu(\mathcal{F} \setminus \mathcal{G}) \leq \frac{c_1}{2} + \frac{c_1}{2} = c_1$  if  $\mu_i^-(\mathcal{G}) \leq \frac{c_1}{2}$ , and  $\mu_i^+(\mathcal{F}) \leq \mu_i^+(\mathcal{G}) + 2\mu(\mathcal{F} \setminus \mathcal{G}) \leq \frac{c_1}{2} + \frac{c_1}{2} = c_1$  if  $\mu_i^+(\mathcal{G}) \leq \frac{c_1}{2}$ . Hence,  $\mathcal{F}$  is good, a contradiction.  $\square$

**7.2. Reduction to the case where  $\mathcal{F}$  is increasing.** Here we show that one can transform  $\mathcal{F}$  into an increasing family by a series of  $\frac{c_1}{2}$ -small modifications. (For brevity, if  $i \in [n]$  we henceforth write  $\mathcal{S}_{\emptyset i}$  for  $\mathcal{S}_{\emptyset\{i\}}$ .) It suffices to prove the following lemma.

**Lemma 7.3.** *Let  $\mathcal{F} \subset \mathcal{P}([n])$  with  $\mu(\mathcal{F}) = \mu$ ,  $\mu_i^-(\mathcal{F}) \leq \mu_i^+(\mathcal{F})$  for all  $i \in [n]$ , and  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_1$ . Then for each  $i \in [n]$ ,  $\mathcal{S}_{\emptyset i}(\mathcal{F})$  is a  $\frac{c_1}{2}$ -small modification of  $\mathcal{F}$ .*

*Proof.* By Lemma 3.4, we have  $I[\mathcal{S}_{\emptyset i}(\mathcal{F})] \leq I[\mathcal{F}]$ . By the isoperimetric inequality, we have

$$(\mu_i^+ - \mu_i^-) + \frac{1}{2}I[\mathcal{L}_{\mu_i^-}] + \frac{1}{2}I[\mathcal{L}_{\mu_i^+}] = I[\mathcal{L}_{\mu_i^-, \mu_i^+}] \geq I[\mathcal{L}_\mu].$$

By Lemma 3.1, and by the isoperimetric inequality applied to  $\mathcal{F}_{\{i\}}^{\{i\}}$  and to  $\mathcal{F}_{\{i\}}^\emptyset$ , we have

$$(7.1) \quad \text{Inf}_i[\mathcal{F}] + \frac{1}{2}I[\mathcal{L}_{\mu_i^-}] + \frac{1}{2}I[\mathcal{L}_{\mu_i^+}] \leq I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_1.$$

These imply that

$$(7.2) \quad \text{Inf}_i[\mathcal{F}] - (\mu_i^+ - \mu_i^-) \leq c_1.$$

Now note that

$$(7.3) \quad \text{Inf}_i[\mathcal{F}] = \mu_i^+(\mathcal{S}_{\emptyset i}(\mathcal{F})) - \mu_i^-(\mathcal{S}_{\emptyset i}(\mathcal{F})).$$

Combining (7.2) and (7.3), we obtain

$$\mu(\mathcal{F} \Delta \mathcal{S}_{\emptyset i}(\mathcal{F})) = \frac{1}{2}(\mu_i^+(\mathcal{S}_{\emptyset i}(\mathcal{F})) - \mu_i^+(\mathcal{F})) + \frac{1}{2}(\mu_i^-(\mathcal{F}) - \mu_i^-(\mathcal{S}_{\emptyset i}(\mathcal{F}))) \leq \frac{c_1}{2},$$

as required.  $\square$

The next corollary follows immediately from Lemmas 7.2 and 7.3.

**Corollary 7.4.** *Let  $\mathcal{F} \subset \mathcal{P}([n])$  with  $\mu(\mathcal{F}) = \mu$ ,  $\mu_i^-(\mathcal{F}) \leq \mu_i^+(\mathcal{F})$  for all  $i \in [n]$  and  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_2$ , let  $c_2 \leq c_1$ , and let*

$$\mathcal{G} := (\mathcal{S}_{\emptyset n} \circ \mathcal{S}_{\emptyset n-1} \circ \cdots \circ \mathcal{S}_{\emptyset 1})(\mathcal{F}).$$

*If the increasing family  $\mathcal{G}$  is good, then so is  $\mathcal{F}$ .*

From now on we assume that  $\mathcal{F}$  is increasing.

**7.3. Proof in the case where  $n$  is small.** We now show that Proposition 7.1 holds in the case where  $n$  is small. In fact, crudely, we have the following.

**Lemma 7.5.** *Suppose that  $n \leq n_0$  and  $\mathcal{F} \subset \mathcal{P}([n])$  with  $\mu(\mathcal{F}) = \mu$  and  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_2$ . Then  $\mathcal{F}$  is weakly isomorphic to  $\mathcal{L}_\mu$ , provided  $c_2 < 2^{-(n_0-1)}$ .*

*Proof.* If  $c_2 < 2^{-(n-1)}$ , then we must have  $I[\mathcal{F}] = I[\mathcal{L}_\mu]$  (note that the influence of any family depending on  $n$  variables is of the form  $\frac{1}{2^{n-1}}$ ). The lemma now follows from the uniqueness part of the isoperimetric inequality.  $\square$

**7.4. Reduction to the case where  $\mathcal{F}$  is  $n$ -stable.** We say that  $\mathcal{F}$  is  $n$ -stable if  $\mathcal{S}_{n,i}(\mathcal{F}) = \mathcal{F}$  for each  $i \in [n-1]$ , and if  $A \cup \{n\} \in \mathcal{F}$  whenever  $A \in \mathcal{F}$ . (As usual, we write  $\mathcal{S}_{i,j}$  for  $\mathcal{S}_{\{i\}\{j\}}$ , for brevity.) Here, we show that if  $\mathcal{F} \subset \mathcal{P}([n])$  is bad and  $n$  is large then there exists a  $\frac{c_1}{2}$ -small modification of  $\mathcal{F}$  that is  $n$ -stable. We need the following well-known lemma.

**Lemma 7.6.** *Let  $i \neq j \in [n]$ . Then  $\mu(\mathcal{S}_{i,j}(\mathcal{F})) = \mu(\mathcal{F})$  and  $I[\mathcal{S}_{i,j}(\mathcal{F})] \leq I[\mathcal{F}]$ .*

We remark that the operator  $\mathcal{S}_{i,j}$  preserves monotonicity, for each  $i \neq j$ . We also need the following crude upper bound on the total influence of lexicographically ordered families.

**Claim 7.7.**  *$I[\mathcal{L}_\mu] \leq 2$  for each  $\mu \in [0, 1]$ .*

*Proof.* We prove the claim by induction on  $n$ . If  $n = 1$  then in fact  $I[\mathcal{L}_\mu] \leq 1$ . We may assume that  $\mu \leq \frac{1}{2}$ , since  $I[\mathcal{L}_{1-\mu}] = I[\mathcal{L}_\mu]$ . Hence, the induction hypothesis implies that

$$I[\mathcal{L}_\mu] = I[\mathcal{L}_{0,2\mu}] = 2\mu + \frac{1}{2}I[\mathcal{L}_{2\mu}] \leq 2.$$

□

The following lemma reduces Proposition 7.1 to the case where  $n$  is stable.

**Lemma 7.8.** *Let  $\mathcal{F} \subset \mathcal{P}([n])$  be an increasing family with  $\mu(\mathcal{F}) = \mu$  and  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + c_2$ , suppose that  $\text{Inf}_n[\mathcal{F}] = \min_{i \in [n]} \text{Inf}_i[\mathcal{F}]$ , and let*

$$\tilde{\mathcal{F}} := \mathcal{S}_{n,n-1}(\cdots \mathcal{S}_{n,2}(\mathcal{S}_{n,1}(\mathcal{F}))).$$

*If  $c_2 < \min\{2^{-(6/c_1-1)}, 1, c_1\}$ , and the  $n$ -stable family  $\tilde{\mathcal{F}}$  is good, then so is  $\mathcal{F}$ .*

*Proof.* By Lemma 7.2, it suffices to show that  $\tilde{\mathcal{F}}$  is a  $\frac{c_1}{2}$ -small modification of  $\mathcal{F}$ . For this, by Lemma 7.6, it suffices to prove that  $\mu(\tilde{\mathcal{F}} \Delta \mathcal{F}) \leq \frac{c_1}{2}$ . The key observation is that

$$(7.4) \quad \mu(\tilde{\mathcal{F}} \Delta \mathcal{F}) = 2\mu(\mathcal{F} \setminus \tilde{\mathcal{F}}) \leq 2\mu(\mathcal{I}_n(\mathcal{F})).$$

Indeed,

$$A \mapsto A \cup \{i_A\},$$

where  $i_A := \min\{i : (A \cup \{i\}) \setminus \{n\} \notin \mathcal{F}\}$ , is an injection from  $\mathcal{F} \setminus \tilde{\mathcal{F}}$  to  $\mathcal{I}_n(\mathcal{F})$ . Now note that

$$(7.5) \quad \mu(\mathcal{I}_n(\mathcal{F})) = \frac{\text{Inf}_n(\mathcal{F})}{2} \leq \frac{I(\mathcal{F})}{2n} \leq \frac{I[\mathcal{L}_\mu] + c_2}{2n}.$$

To complete the proof of the lemma, note that (7.4) and (7.5) imply that

$$\mu(\tilde{\mathcal{F}} \Delta \mathcal{F}) \leq \frac{I[\mathcal{L}_\mu] + c_2}{n} < \frac{3}{n},$$

provided  $c_2 < 1$ , using Claim 7.7. Suppose that  $\mathcal{F}$  is bad. By Lemma 7.5, we may assume that  $n \geq 6/c_1$ , provided  $c_2 < 2^{-(6/c_1-1)}$ , and therefore  $\mu(\tilde{\mathcal{F}} \Delta \mathcal{F}) \leq \frac{1}{2}c_1$ . Hence,  $\tilde{\mathcal{F}}$  is a  $\frac{c_1}{2}$ -small modification of  $\mathcal{F}$ , and we are done by Lemma 7.2, provided  $c_2 \leq c_1$ . □

From now on we assume also that  $\mathcal{F}$  is  $n$ -stable.

**7.5. The case where  $|\mathcal{I}_n| \geq 2$ .** Here we show that if  $|\mathcal{I}_n| \geq 2$ , then there exists a  $\frac{1}{2^{n-1}}$ -small modification of  $\mathcal{F}$  with smaller  $|\mathcal{I}_n|$ . This will allow us to reduce to the case where  $|\mathcal{I}_n| \leq 1$ .

**Lemma 7.9.** *Let  $A, B \in \mathcal{I}_n(\mathcal{F})$  such that  $A \neq B$ , and let  $\mathcal{F}_1 = (\mathcal{F} \setminus \{A\}) \cup \{B \setminus \{n\}\}$ ,  $\mathcal{F}_2 = (\mathcal{F} \setminus \{B\}) \cup \{A \setminus \{n\}\}$ . Then either  $I[\mathcal{F}_1] < I[\mathcal{F}]$  or  $I[\mathcal{F}_2] < I[\mathcal{F}]$ .*

*Proof.* Suppose w.l.o.g. that  $|A| \geq |B|$ ; we will show that  $I[\mathcal{F}_1] < I[\mathcal{F}]$ . Since  $\mathcal{F}$  is  $n$ -stable, we have  $A \setminus \{i\} \notin \mathcal{F}$  for each  $i \in A$  (note that  $((A \setminus \{i\}) \setminus \{n\}) \cup \{i\} = A \setminus \{n\} \notin \mathcal{F}$ ). This implies that

$$|\partial(\mathcal{F} \setminus \{A\})| - |\partial\mathcal{F}| \leq n - 2|A|.$$

Since  $(B \setminus \{n\}) \cup \{i\} \in \mathcal{F}$  for each  $i \notin B \setminus \{n\}$ , we have

$$|\partial((\mathcal{F} \setminus \{A\}) \cup \{B \setminus \{n\}\})| - |\partial(\mathcal{F} \setminus \{A\})| \leq 2|B| - n - 2.$$

This implies that  $|\partial\mathcal{F}_1| \leq |\partial\mathcal{F}| + (n - 2|A|) + (2|B| - n - 2) \leq |\partial\mathcal{F}| - 2$ .  $\square$

**7.6. Proof of Proposition 7.1.** We prove the proposition by induction on  $n$ . Let  $c_1 = 2^{-k}$ , where  $k \in \mathbb{N}$ . Assume that  $c_2 < \min\{2^{-(6/c_1-1)}, 1, c_1\}$ . The case  $n \leq k+1$  follows from Lemma 7.5. Let  $n \geq k+2$ , and let  $\mathcal{F}$  be as in the hypothesis of the proposition. Suppose for a contradiction that  $\mathcal{F}$  is bad. By Corollary 7.4, we may assume that  $\mathcal{F}$  is increasing; by Lemma 7.8, we may assume that  $\mathcal{F}$  is  $n$ -stable, and by Lemmas 7.2, 7.5, and 7.9 we may assume that  $|\mathcal{I}_n(\mathcal{F})| \leq 1$ . If  $|\mathcal{I}_n(\mathcal{F})| = 0$ , then  $\mathcal{F}$  does not depend on the  $n$ th coordinate and the proposition holds by the induction hypothesis. Suppose that  $|\mathcal{I}_n(\mathcal{F})| = 1$ . Then, by Lemma 3.1, we have

$$(7.6) \quad I[\mathcal{F}] = \frac{1}{2}I\left[\mathcal{F}_{\{n\}}^{\{n\}}\right] + \frac{1}{2}I\left[\mathcal{F}_{\{n\}}^{\emptyset}\right] + \text{Inf}_n[\mathcal{F}] = \frac{1}{2}I\left[\mathcal{F}_{\{n\}}^{\{n\}}\right] + \frac{1}{2}I\left[\mathcal{F}_{\{n\}}^{\emptyset}\right] + \frac{1}{2^{n-1}}.$$

Since  $|\mathcal{F}|$  is odd, and since in the lexicographic ordering, sets containing  $n$  alternate with sets not containing  $n$ , we have  $\mathcal{L}_\mu = (\mathcal{L}_{\mu_n^-, \mu_n^+})_{\text{ass}}$ , and therefore

$$(7.7) \quad I[\mathcal{L}_\mu] = \frac{1}{2}I\left[\mathcal{L}_{\mu_n^-(\mathcal{F})}\right] + \frac{1}{2}I\left[\mathcal{L}_{\mu_n^+(\mathcal{F})}\right] + \frac{1}{2^{n-1}}.$$

By (7.6) and (7.7), we either have  $I\left[\mathcal{F}_{\{n\}}^{\{n\}}\right] \leq I\left[\mathcal{L}_{\mu_n^+}\right] + c_2$  or  $I\left[\mathcal{F}_{\{n\}}^{\emptyset}\right] \leq I\left[\mathcal{L}_{\mu_n^-}\right] + c_2$ . Recall that we are assuming  $c_1 = 2^{-k}$  for some  $k \in \mathbb{N}$ , and that  $n \geq k+2$ . Since  $1/4 + 2^{-k} \leq \mu = \mu_n^- + 2^{-n}$ ,  $2^{n-1}\mu_n^- \in \mathbb{Z}$  and  $n > k$ , we have  $\mu_n^- \geq 1/4 + 2^{-k}$ . Moreover, since  $\mu \leq 1/2$ ,  $\mu_n^+ = \mu + 2^{-n}$  and  $2^{n-1}\mu_n^+ \in \mathbb{Z}$ , we must have  $\mu \leq 1/2 - 2^{-n}$ , and therefore  $\mu_n^+ \leq 1/2$ . Hence,

$$\frac{1}{4} + c_1 \leq \mu_n^- < \mu_n^+ \leq \frac{1}{2}.$$

Therefore, we may apply the induction hypothesis to one of  $\mathcal{F}_{\{n\}}^{\emptyset}$  and  $\mathcal{F}_{\{n\}}^{\{n\}}$ . Suppose first that  $\mu_i^-\left(\mathcal{F}_{\{n\}}^{\emptyset}\right) \leq c_1$ . Then, by Lemma 6.1, we have  $\mu_i^-\left(\mathcal{F}_{\{n\}}^{\emptyset}\right) \leq \frac{c_2}{2}$ . This implies that

$$\mu_i^-(\mathcal{F}) \leq \mu_i^-\left(\mathcal{F}_{\{n\}}^{\emptyset}\right) + 2^{-n} \leq \frac{1}{2}c_2 + 2^{-n} \leq c_1 = 2^{-k}$$

since  $c_2 \leq c_1$  and  $n > k$ . Hence,  $\mathcal{F}$  is good, as desired. The case where  $I\left[\mathcal{F}_{\{n\}}^{\{n\}}\right] \leq I\left[\mathcal{L}_{\mu_n^+(\mathcal{F})}\right] + c_2$  is similar.

8.  $\mathcal{F}$  IS ESSENTIALLY CONTAINED IN A CODIMENSION-2 SUBCUBE ( $\mu$  SMALL)

In this section we essentially complete the proof of Proposition 4.1 in the case where  $\mu(\mathcal{F})$  is ‘small’. Specifically, we prove the following.

**Proposition 8.1.** *For each  $c > 0$ , there exists  $d = d(c) > 0$  such that the following holds. Suppose  $\mathcal{F} \subset \mathcal{P}([n])$  with  $\mu := \mu(\mathcal{F}) \leq \frac{1}{4} + d$  and  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + d\mu$ . Then there exists a family  $\mathcal{G}$  weakly isomorphic to  $\mathcal{F}$ , such that  $\mu_1^-(\mathcal{G}) \leq \mu_2^-(\mathcal{G}) \leq c\mu$ .*

We start by reducing to the case where  $\mathcal{F}$  is increasing.

**Lemma 8.2.** *If Proposition 8.1 holds for all increasing families  $\mathcal{F}$ , then it holds for all families  $\mathcal{F}$ .*

*Proof.* Given  $c > 0$ , let  $d' = d'(c) > 0$  such that each increasing family  $\mathcal{G}$  with  $\mu(\mathcal{G}) \leq 1/4 + d'$ , satisfying  $I[\mathcal{G}] \leq I[\mathcal{L}_{\mu(\mathcal{G})}] + d'\mu(\mathcal{G})$  also satisfies  $\mu_i^-(\mathcal{G}) \leq \mu_j^-(\mathcal{G}) \leq \frac{c\mu(\mathcal{G})}{2}$  for some  $i \neq j \in [n]$ . Let  $d = \min\{c, d'\}$ . Let  $\mathcal{F} \subset \mathcal{P}([n])$  be some family satisfying  $\mu := \mu(\mathcal{F}) \leq 1/4 + d$  and  $I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + d\mu$ . We may assume without loss of generality that  $\mu_i^-(\mathcal{F}) \leq \mu_i^+(\mathcal{F})$  for each  $i \in [n]$ . By Lemma 7.3, we have  $\mu(\mathcal{S}_{\emptyset 1}(\mathcal{F}) \Delta \mathcal{F}) \leq \frac{d\mu}{2}$ , and therefore

$$(8.1) \quad \mu(\mathcal{F}_{\{1\}}^\emptyset) \leq \mu((\mathcal{S}_{\emptyset 1}(\mathcal{F}))_{\{1\}}^\emptyset) + \frac{1}{2}d\mu.$$

Note also that  $\mu(\mathcal{S}_{\emptyset 1}(\mathcal{F})_{\{i\}}^\emptyset) = \mu(\mathcal{F}_{\{i\}}^\emptyset)$  for any  $i \geq 2$ . Now let

$$\mathcal{G} := \mathcal{S}_{\emptyset n}(\mathcal{S}_{\emptyset(n-1)} \cdots (\mathcal{S}_{\emptyset 1}(\mathcal{F}))) ;$$

clearly,  $\mathcal{G}$  is increasing. As in (8.1), we have

$$\mu(\mathcal{F}_{\{i\}}^\emptyset) \leq \mu(\mathcal{G}_{\{i\}}^\emptyset) + \frac{1}{2}d\mu \quad \forall i \in [n].$$

By Lemma 3.4, we have  $I[\mathcal{G}] \leq I[\mathcal{F}] \leq I[\mathcal{L}_\mu] + d\mu$ . Since  $d \leq d'$ , there exist two coordinates  $i \neq j \in [n]$  such that  $\mu_i^-(\mathcal{G}) \leq \mu_j^-(\mathcal{G}) \leq \frac{c\mu}{2}$ . Hence,  $\mu_i^-(\mathcal{F}) \leq \mu_i^-(\mathcal{G}) + \frac{d\mu}{2} \leq c\mu$ , and similarly,  $\mu_j^-(\mathcal{F}) \leq \mu_j^-(\mathcal{G}) + \frac{d\mu}{2} \leq c\mu$ , as required.  $\square$

The key lemma for the proof of Proposition 8.1 is the following.

**Lemma 8.3.** *Let  $\mathcal{F} \subset \mathcal{P}([n])$  be increasing, with  $\mu(\mathcal{F}) \leq \frac{1}{2}$ . Let*

$$\mathcal{F}_1 = \mathcal{S}_{n,1} \mathcal{S}_{n-1,1} \circ \cdots \circ \mathcal{S}_{2,1}(\mathcal{F}),$$

$$\mathcal{F}_2 = \mathcal{S}_{\{n,n-1\}1} \circ \cdots \circ \mathcal{S}_{\{3,2\}1}(\mathcal{F}_1),$$

$$\vdots$$

$$\mathcal{F}_n = \mathcal{S}_{\{n,n-1,\dots,2\}1}(\mathcal{F}_{n-1}).$$

Then

- (i)  $\mathcal{F}_n$  is contained in the dictatorship  $\mathcal{D}_1$ ,
- (ii)  $I[\mathcal{F}_n] \leq I[\mathcal{F}]$ , and
- (iii)  $\mu_i^-(\mathcal{F}_n) \geq \mu_i^-(\mathcal{F})$  for any  $i > 1$ .

*Proof.* To prove (i), first note that  $\mathcal{F}_n$  is increasing, using the fact that  $\mathcal{S}_{S_1}(\mathcal{G})$  is increasing whenever  $\mathcal{G}$  is increasing and  $\mathcal{S}_{S'_1}(\mathcal{G}) = \mathcal{G}$  for all  $S' \subset S$  with  $|S'| = |S| - 1$ . Suppose for a contradiction that  $\mathcal{F}_n \not\subseteq \mathcal{D}_1$ ; then there exists  $S \subset \{2, \dots, n\}$  such that  $S \in \mathcal{F}_n$ , so by the monotonicity of  $\mathcal{F}_n$ , we have  $\{2, 3, \dots, n\} \in \mathcal{F}_n$ . But then, by construction of  $\mathcal{F}_n$ , we have  $\mathcal{D}_1 \subset \mathcal{F}_n$ , and so  $\mathcal{D}_1 \cup \{\{2, 3, \dots, n\}\} \subset \mathcal{F}_n$ , contradicting the fact that  $\mu(\mathcal{F}_n) = \mu(\mathcal{F}) \leq 1/2$ .

Statement (ii) follows by repeated application of Lemma 3.3, and (iii) is clear.  $\square$

The idea of the proof of Proposition 8.1 is as follows. Let  $\mathcal{F} \subset \mathcal{P}([n])$  be an increasing family as in the hypothesis of the proposition; assume w.l.o.g. that  $\mu_1^-(\mathcal{F}) = \min_i (\mu_i^-(\mathcal{F}))$ . Let  $\mathcal{F}_n$  be the family from Lemma 8.3. By Lemma 8.3, we have  $\mu_i^-((\mathcal{F}_n)_{\{1\}}^{\{1\}}) = 2\mu_i^-(\mathcal{F}_n) \geq 2\mu_i^-(\mathcal{F})$  for all  $i > 1$ , and  $\mu((\mathcal{F}_n)_{\{1\}}^{\{1\}}) = 2\mu(\mathcal{F})$ . This allows us perform an inductive argument, doubling the measure of the family at each step, and thus reducing to the case of measure somewhat larger than  $1/4$  (encapsulated in the following lemma, which enables us to do the base case of the induction).

**Lemma 8.4.** *For each  $c > 0$ , there exist  $d_1 = d_1(c) > 0$ ,  $d_2 = d_2(c) > 0$  such that the following holds. Suppose that  $\mathcal{F} \subset \mathcal{P}([n])$  is increasing with  $\frac{1}{4} + d_1 \leq \mu(\mathcal{F}) \leq \frac{1}{2} + 2d_1$ , and that*

$$I[\mathcal{F}] \leq I[\mathcal{L}_{\mu(\mathcal{F})}] + d_2\mu(\mathcal{F}).$$

*Then  $\mu_i^-(\mathcal{F}) \leq c\mu(\mathcal{F})$  for some  $i \in [n]$ .*

*Proof.* If  $\mu(\mathcal{F}) \leq \frac{1}{2}$ , the lemma follows from applying Proposition 7.1 to  $\mathcal{F}$ . In the case where  $\mu(\mathcal{F}) \geq \frac{1}{2}$ , it follows by applying Proposition 7.1 to  $\mathcal{F}^c$ .  $\square$

We now prove Proposition 8.1.

*Proof of Proposition 8.1.* By Lemma 8.2, it suffices to prove the proposition for increasing families. Let  $c > 0$ , and let  $\mathcal{F} \subset \mathcal{P}([n])$  be an increasing family as in the statement of the proposition, where  $d = \min\{d_1(c), d_2(c)\}$  and  $d_1(c), d_2(c)$  are as in Lemma 8.4. Write  $\mu := \mu(\mathcal{F})$ . Write  $\mu = (\frac{1}{2})^j \cdot \mu_0$ , where  $\frac{1}{4} + d_1 \leq \mu_0 \leq \frac{1}{2} + 2d_1$  and  $j \in \mathbb{N}$ . We prove by induction on  $j$  that Proposition 8.1 holds (for increasing families) with the above choice of  $d$ . Let  $j \geq 1$ . Suppose w.l.o.g. that  $\mathcal{F}$  satisfies

$$\mu_1^-(\mathcal{F}) \leq \mu_2^-(\mathcal{F}) \leq \dots \leq \mu_n^-(\mathcal{F}).$$

Let  $\mathcal{F}_n$  be as in Lemma 8.3. Let  $\mathcal{F}' = (\mathcal{F}_n)_{\{1\}}^{\{1\}}$ ; then  $\mu(\mathcal{F}') = 2\mu$ . If  $j = 1$ , then  $\mu(\mathcal{F}') = \mu_0 \in [1/4 + d_1, 1/2 + 2d_1]$ , and so it follows from Lemma 8.4 that

$$\min_{i \geq 1} \mu_i^-(\mathcal{F}') \leq c\mu(\mathcal{F}') = 2c\mu.$$

Suppose that  $\min_{i \geq 1} \mu_i^-(\mathcal{F}') = \mu_m^-(\mathcal{F}')$ . Then

$$\mu_1^-(\mathcal{F}) \leq \mu_2^-(\mathcal{F}) \leq \mu_m^-(\mathcal{F}) \leq \mu_m^-(\mathcal{F}_n) = \frac{1}{2}\mu_m^-(\mathcal{F}') \leq c\mu,$$

completing the base case of the induction. Now let  $j \geq 2$ , and assume the desired statement holds when  $j$  is replaced by  $j - 1$ . Applying the induction hypothesis to  $\mathcal{F}'$  yields

$$\min_{i \geq 1} \mu_i^-(\mathcal{F}') \leq c\mu(\mathcal{F}') = 2c\mu,$$

and so by the same argument as above, we have  $\mu_1^-(\mathcal{F}) \leq \mu_2^-(\mathcal{F}) \leq c\mu$ , completing the inductive step, and proving the proposition.  $\square$

## 9. WRAPPING UP THE PROOF OF PROPOSITION 4.1

Proposition 4.1 follows easily by combining Propositions 7.1 and 8.1 with the corresponding bootstrapping lemmas. We recall the statement of Proposition 4.1 for the convenience of the reader.

**Proposition.** *There exist absolute constants  $c_1, c_2 > 0$  such that the following holds. Let  $\mu \leq \frac{1}{2}$ , let  $\epsilon \leq c_1 \mu$ , and let  $\mathcal{F} \subset \mathcal{P}([n])$  be a family with  $I[\mathcal{F}] = I[\mathcal{L}_\mu] + \epsilon$  and  $\mu(\mathcal{F}) = \mu$ . Then there exists a family  $\mathcal{G}$  weakly isomorphic to  $\mathcal{F}$  such that one of the following holds.*

- *Case (1):  $c_2 \mu_1^-(\mathcal{G}) + \frac{1}{2} \epsilon_1^+(\mathcal{G}) \leq \epsilon$ , or*
- *Case (2):  $c_2 \mu(\mathcal{G} \setminus \mathcal{S}_{\{1,2\}}) + \frac{1}{4} \epsilon_{1,2}^{++}(\mathcal{G}) \leq \epsilon$ .*

*Proof.* Let  $\mathcal{F} \subset \mathcal{P}([n])$  be as in the hypothesis of the proposition. Let  $\mathcal{G}$  be a family weakly isomorphic to  $\mathcal{F}$ , satisfying

$$\mu_1^-(\mathcal{G}) \leq \mu_2^-(\mathcal{G}) \leq \cdots \leq \mu_n^-(\mathcal{G}) \leq \mu_n^+(\mathcal{G}) \leq \cdots \leq \mu_1^+(\mathcal{G}).$$

Lemma 6.2 implies that either Case (1) or Case (2) holds if  $\mu_2^-(\mathcal{G}) \leq \frac{1}{6} \mu$ , provided  $c_2 \leq \frac{1}{6}$ . By Proposition 8.1, there exists  $d > 0$  such that the inequality  $\mu_2^-(\mathcal{G}) \leq \frac{1}{6} \mu$  holds provided  $\mu(\mathcal{G}) \leq \frac{1}{4} + d$ , and provided  $c_1 \leq d$ . Thus, by Lemma 6.2, Case (1) or Case (2) holds for any family  $\mathcal{F}$  satisfying  $\mu(\mathcal{F}) \leq \frac{1}{4} + d$ .

We may henceforth assume that  $\frac{1}{4} + d \leq \mu(\mathcal{F}) \leq \frac{1}{2}$ . By Proposition 7.1, we have  $\mu_1^-(\mathcal{G}) \leq d$  provided  $c_1$  is sufficiently small depending on  $d$ . Hence, by Lemma 6.1, we have  $c_2 \mu_1^-(\mathcal{G}) + \frac{1}{2} \epsilon_1^+(\mathcal{G}) \leq \epsilon$ , provided  $c_2 \leq 2$ , so Case (1) holds. This completes the proof.  $\square$

**Remark 9.1.** For ease of exposition, we have not attempted to optimize the value of the absolute constant  $C$  given by our proof of Theorem 1.5. (It can be checked that our proof, as written, yields  $C = 2^{6 \cdot 2^{360}}$ . This can easily be reduced to  $C = 2^{360}$ , by introducing an extra constant into the statement of Proposition 8.1.) Unfortunately, it does not seem possible to modify our approach to obtain  $C = 2$  (see Conjecture 10.1 below).

## 10. CONCLUSION AND OPEN PROBLEMS

As mentioned above, we conjecture that Theorem 1.5 holds with  $C = 2$ .

**Conjecture 10.1.** *If  $\mathcal{F} \subset \mathcal{P}([n])$  and  $\mathcal{L} \subset \mathcal{P}([n])$  is the initial segment of the lexicographic ordering with  $|\mathcal{L}| = |\mathcal{F}|$ , then there exists a family  $\mathcal{G} \subset \mathcal{P}([n])$  weakly isomorphic to  $\mathcal{L}$ , such that*

$$|\mathcal{F} \Delta \mathcal{G}| \leq 2(|\partial \mathcal{F}| - |\partial \mathcal{L}|).$$

More generally, it would be of interest to determine more precisely the behaviour of the function

$$s(n, m, l) := \max\{\min\{|\mathcal{F} \Delta \mathcal{G}| : \mathcal{G} \cong \mathcal{L}\} : \mathcal{F} \subset \mathcal{P}([n]), |\mathcal{F}| = m, |\partial \mathcal{F}| \leq |\partial \mathcal{L}| + l\},$$

where  $\mathcal{L}$  denotes the initial segment of the lexicographic ordering of size  $m$ .

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